Building Reputation at the Edge of the Cliff

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Abstract

We present a bargaining model with incomplete information and players who repeatedly revise their actions over a finite period of time. For an open set of parameters, one-sided reputation-building leads to a deadline effect in the form of last-minute strategic interaction, and is linked with a non-negligible risk of reaching an inefficient outcome. Two-sided reputation-formation necessarily induces substantial delay and inefficiency with positive probability.

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The 2012 New Year's Eve celebrations in America were somewhat clouded by the gloomy predictions of the Congressional Budget Office about an upcoming fiscal crisis that might take place in 2013. This unsettling forecast originated from the expiration of several laws, most notably the 2010 Tax Relief Act and the Budget Control Act of 2011, which entailed an increase

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in taxes as well as major spending cuts, leading to a sharp decline in the budget deficit. If all the changes were to go into effect simultaneously they would have induced a recession by cutting household incomes, increasing unemployment rates, and hurting both consumers' and investors' confidence in the economy. This dire situation sparked extensive media coverage that referred to the December 31 midnight deadline and the sharp decline in the budget deficit expected to ensue as the "fiscal cliff."

Preventing the fiscal cliff was supposedly a very simple task. Either the tax reliefs were to be extended, spending cuts were to be canceled, or some combination of these measures was to be taken. However, the political situation provided an extremely inconvenient environment for enacting such reforms. President Barack Obama and the Democratic-controlled Senate disapproved of across-the-board tax cuts (as opposed to tax cuts for only the bottom 98%), and wanted to keep the spending level relatively high. The Republican-controlled House of Representatives preferred a solution that would lower spending as well as tax rates. Several proposals for amending the budget had been suggested by President Obama, House Speaker John Boehner, and others, but all were quickly rejected. In an attempt to reach an agreement, negotiations extended until the very last hours of 2012. There was some uncertainty about whether a compromise would be reached in time, and the entire bipartisan negotiation process was described by several commentators as an elaborate and dangerous high-stakes game of "chicken."[1] An agreement was finally reached just before the deadline, with legislation passing in the Senate on January 1, and in the House the following day.[2]

This paper models the negotiations as a revision game with reputation formation.[3] In the revision game model, introduced by Kamada and Kandori (2011), players prepare (pure) actions over a continuous and finite time horizon. They can change their actions only when they are called to play by

[1]See, for example, Robin Harding, "US Plays Chicken on Edge of Fiscal Cliff", ft.com/world, November 11, 2012, and Robert Reich, "Cliff Hanger: Obama's Last Stand and the Republican Strategy of Fanaticism", Huffington Post, December 26, 2012.

[2]Several economists insist that the consequences of passing the deadline by several days would not have been as catastrophic as portrayed by the media; see, e.g., Baker (2013). However, our theoretical analysis applies to other situations of negotiating close to a deadline, as well as to the described situation where the utilities of the players represent the political cost of not reaching agreement for the two parties rather than the actual economic implications of the fiscal cliff.

[3]Our results also apply to a model of war of attrition with a deadline and with independent and stochastically distributed exit opportunities (see Section 4).

stochastic Poisson processes. When the deadline is reached, the last actions prepared are used to determine the payoffs. This model also encompasses the idea of an uncertain deadline, as its effect is similar to the randomness induced by the stochastic revision opportunities.[4] We expand the model to accommodate for incomplete information, which allows us to study how adding a small probability irrational type into the game affects the equilibrium outcome.

In simplifying the negotiation alternatives to a 2x2 game of opposing interests, we demonstrate that the effect of reputation-building on equilibrium outcomes can be substantial, even as the time horizon becomes infinitely long. That is, we show that, generically, reputation formation by one player prevents her opponent from achieving her most-preferred outcome (Proposition 1). Furthermore, one-sided reputation-building often leads to last-minute revisions that push some of the strategic interaction close to the deadline and induce a chance of falling over the cliff, i.e., reaching an outcome that neither side desires (Theorem 2).[5] When both parties try to build reputation, substantial delay must arise with positive probability and inefficiency is inevitable (Theorem 4). Furthermore, in this case the probability of not reaching an agreement can be non-negligible. It is important to stress that as our model contains no flow payoffs inefficiencies are caused only by ex-post Pareto inefficient outcomes, and that as a result of players' inability to revise their action continuously any form of delayed action is necessarily tightly connected with inefficiency.

We provide some illustrative comparative statistics, as well as suggestive computational evidence, that help in assessing the magnitude of inefficiency. These results demonstrate that the more players are similar in strength, the more likely they are to hold to their bargaining position for a long time, leading to a deadline effect with harmful implications for the players' expected utility. In the limit case of equal strengths, the inefficiency does not vanish even as the ex-ante probability of the commitment types approaches zero.

Some methods used in the study of revision games are similar to those employed when discussing wars of attrition. We introduce a model of war of attrition over a continuous and finite horizon with Poisson arrivals and

[4] Explicitly adding a deadline over which there is common uncertainty would not substantially change any of the results.

[5] It is straightforward to show existence of an equilibrium with such properties. Our contribution is in showing that every sequential equilibrium *must* exhibit the same inefficiency. The same is true for most of the results in this paper.

incomplete information. We prove that all sequential equilibria possess a simple structure in which one of the players uses a strategy that is completely characterized by a cutoff time, and the other player's strategy also adheres to the same cutoff time (but could be more complicated before it). This allows us to provide sharper results concerning delay and inefficiency in one-sided and two-sided reputation-building scenarios.

The deadline effect that our model predicts is widely prevalent not only in political circumstances, but also in situations of a purely economic nature. It was observed in empirical data on labor strikes (Cramton and Tracy, 1992), and was replicated in lab experiments (Roth *et al.*, 1988). While the present paper stresses the role of reputation-building in inducing such an effect, other authors have suggested explanations such as irreversible commitments (Fershtman and Seidmann, 1993), private information about second-order beliefs (Feinberg and Skrzypacz, 2005), strategic delay in a bargaining process (Ma and Manove, 1993), individual deadlines (Sandholm and Vulkan, 1999), and optimism (Yildiz, 2004). Ponsati (1995) discusses continuous time-bargaining with private information and a deadline and points to a positive mass of agreements at the deadline, which is hard to interpret as inefficiency without a proper discretization. Fuchs and Skrzypacz (2013) study a scenario with evenly spaced opportunities in which a seller and a buyer can trade, where the buyer has private information over the value of the good and bargaining takes place until a given deadline. They show that there is a deadline effect (an atom of trading probability) right at the deadline, specifically as the gap between trading opportunities goes to zero. In a recent working paper, Fanning (2013) explores a similar theme in an alternating offers game with a deadline and commitment types.

This paper belongs to a growing body of work that follows Kamada and Kandori (2011) and studies different aspects of revision games. Some of the intuition is based on the analysis of the complete-information case by Calcagno *et al.* (2014). Other papers that employ similar methods include Ishii and Kamada (2011) and Kamada and Muto (2011). A few papers use a continuous-time finite-horizon framework to explore specific topics such as bargaining (Ambrus and Lu, 2010) or online auctions (Ambrus *et al.*, 2013). The novel feature of this strand of the literature is the tractable compromise between having a continuous-time model, which often requires extreme technical effort to get rid of unwanted equilibria, and using a discrete-time model, which may add additional strategic aspects (such as the exact order of play) that have nothing to do with the main issues.[6]

This work also relates to a vast literature that explores the effects of building reputation in repeated games, following the seminal papers of Kreps and Wilson (1982a) and Milgrom and Roberts (1982). Several of the many papers on reputation, and perhaps most notably Abreu and Gul (2000), demonstrate the existence of delay and inefficiency, where delay is measured from the first period of play and inefficiency is due to forgone payoff opportunities. Our results resemble these contributions in the sense that perturbing the game by adding commitment types can significantly shift the outcome when the time horizon is long enough. Players sacrifice utility to convince their opponent of their intentions not by decreasing immediate payoffs (as in the repeated games literature), but rather by increasing the probability of reaching an inferior outcome at the deadline. In this paper, the delay is measured backward from the deadline, and reputation formation is based on reaching an ex-post inefficient outcome with positive probability.[7]

The war of attrition extension is reminiscent of Fudenberg and Tirole (1986) who study what happens in a duopoly exit situation when there are commitment types. Unlike our model which is in continuous-time but has discrete preparation opportunities, in their model players can exit at any point in time, which makes their uniqueness proof far more involved. Other related models are the continuous-time model with two-sided uncertainty of Kreps and Wilson (1982a, Section 4),[8] the bargaining model of Osborne (1985), the discrete time model with generalized reputation of Kornhauser *et al.* (1989), and Atakan and Ekmekci (2013) who show that equilibrium behavior in a repeated game with two-sided reputation-building is similar to a war of attrition.

The rest of the paper is organized as follows. Section 1 introduces the model and the formal notation. Section 2 analyzes the general form in the presence of one-sided reputation. Section 3 deals with the case of two-sided

[6] For a relevant discussion on this topic see Hendricks *et al.* (1988).

[7] It is possible to apply some involved transformations between the revision game model and the repeated game model, so that the probabilities of reaching a Pareto inferior outcome are reflected in flow payoffs and in the discount factor. Nevertheless, using any such transformation will strip the model, the methods, and the results of any reasonably intuitive interpretation. We choose not to pursue the search for such an equivalence any further in this paper.

[8] That model uses flow payoffs and strategies in continuous-time, and therefore has a recursive structure.

		Republicans	
		(L)arger spending	(R)educed spending
Democrats	Taxes (U)p	$u_1(U,L), u_2(U,L)$	$u_1(U,R), u_2(U,R)$
	Taxes (D)own	$u_1(D,L), u_2(D,L)$	$u_1(D,R), u_2(D,R)$

Figure 1: Payoff matrix for rational types

reputation-building and provides results concerning the induced inefficiency. Section 4 discusses wars of attrition with a deadline, and Section 5 concludes. Proofs are relegated to the appendix.

1 Model

Since the fiscal cliff negotiations included many elements and neither side had full control over any of the parameters of the final proposal, we choose here to abstract away from the details and present a simplified version of the process. We consider a two-player Bayesian revision game,[9] summarized by the parameters $(T; u_1, u_2; \xi_1, \xi_2)$. For $i \in \{1,2\}$, $u_i \colon \{U,D\} \times \{L,R\} \to \mathbb{R}$ is a payoff function for the rational type of Player i (see Figure 1 below), and $\xi_i \in [0,1]$ is the probability that Player i is a commitment type. We refer to the normal form game associated with the payoff matrix in Figure 1 as the *component game*.

The underlying story is that Player 1 (the Democratic Party) has sole authority to set taxes back to their original level (before the expiration of the tax relief acts), whereas Player 2 (the Republican Party) alone can approve a spending increase. We assume that both (U,L) and (D,R) are strict Nash equilibria of the component game and that

$$u_1(U,L) > u_1(D,R) \text{ and } u_2(U,L) < u_2(D,R).$$

These assumptions represent two main features of the story. First, the game is a game of opposing interests; i.e., each party has a preferred solution to the fiscal crisis situation.[10] The Democrats favor raising taxes and keep

[9]Alternatively, we could have used the model of Kamada and Kandori (2011) with a different equilibrium concept that takes into account the evolving probabilities about (nonexistent) irrational types.

[10]We do not consider common interest games since the analysis of such games does

spending high, while the Republicans would rather lower taxes and embrace the upcoming spending cuts. Second, if both parties insist on their preferred outcome, then the economy falls over the fiscal cliff. That is, if the Democrats keep taxes up and the Republicans go with the reduced spending, then the resulting scenario is undesirable for both players compared to any agreed-upon solution. Finally, if both parties concede then the combination of low taxes and high spending may drive the economy to a fiscal wall, a situation in which there is too much spending and not enough taxes to cover it, and this alternative is unattractive for both players as well.

Denote the possible types of Player i by τ_i^r (rational type) and τ_i^c (commitment type). We model the commitment type of Player 1 (Player 2) as an agent who can only prepare U (R). We often refer to U and R as the commitment actions. We stress that the rational types' payoffs are independent of their rival's type.

Following the notation used by Calcagno *et al.* (2014), players prepare actions on the interval $[-T, 0]$, and the component game is played once at time 0. At time $-T$, both players simultaneously choose the initial profile of actions. We restrict players to choose pure actions at this initial choice.[11] Between time $-T$ and 0, Players 1 and 2 are called to prepare an action according to two independent Poisson processes. For expositional purposes, we assume in the main text that the frequencies of the Poisson processes are both equal to 1; i.e., each player has on average one revision opportunity per unit of time. All the results for arbitrary revision rates are stated and proved in the appendix. Players are not informed of the realizations of their opponent's Poisson process, but are informed of the current profile at any point in time.[12] This means that players' strategies depend only on their

not become more interesting with the introduction of reputation. In such games rational players have no incentive to communicate anything other than being rational and willing to cooperate.

[11]This restriction is mostly for clarity of exposition and focusing the analysis on what we perceive as the important part of the game, namely, the revision phase. In an earlier draft we show that most of the results continue to hold if we allow mixed action at $-T$. We can also replace this assumption by an exogenous selection of a default profile of actions (U, R).

[12]Loosely interpreted: one player cannot tell when the other player considered preparing a different action, but only when the opponent actually did prepare a different action. It is also possible to consider a model in which players are aware of opponents' revision times. However, analyzing this model becomes cumbersome very quickly due to discontinuities at the revision opportunities. See also the concluding discussion.

own preparation opportunities, the prepared action profiles, and time, but not on their opponents' preparation opportunities. At $t = 0$ the action profile that has been prepared most recently determines the payoffs for the players. In order for expected payoffs and probabilities of preparing certain profiles in the revision phase to be well defined for all strategy profiles, we restrict players' strategies to be measurable with respect to the natural topologies. All the elements of the model are common knowledge, and the type of each player is private information (see the appendix for a formal definition of the strategy space).

We employ the solution concept of Sequential Equilibrium (SE) (Kreps and Wilson, 1982b), which guarantees that even off the equilibrium path a player that revised her action to D or L is believed to be rational with probability 1.[13] We consider the limit set of SE payoffs as the length of time horizon, T, approaches infinity.[14] We are interested mostly in the rational types' payoffs, and so we denote by $\phi(T; u_1, u_2; \xi_1, \xi_2)$ the set of interim SE payoffs of the profile (τ_1^r, τ_2^r). We define the revision equilibrium payoff set of $(u_1, u_2; \xi_1, \xi_2)$ by

$$\bar{\phi}(u_1, u_2; \xi_1, \xi_2) = \limsup_{T' \to \infty} \phi(T'; u_1, u_2; \xi_1, \xi_2).$$

Besides the eventual payoffs of the players, we also wish to discuss cases in which a significant delay in reaching an agreement is mandated by equilibrium behavior. We distinguish between two situations in which strategic interaction is significantly delayed. We say that a vector of parameters $(u_1, u_2; \xi_1, \xi_2)$ induces substantial delay if with strictly positive probability the prepared profile does not change throughout the game until close to the deadline. We say that it exhibits last-minute strategic interaction if there is a strictly positive probability that the prepared profile changes close to the deadline. In both cases, we say that the vector of parameters induces inefficiency if there is a strictly positive probability of reaching ex-post Pareto inefficient payoffs. The following definition formalizes these statements.

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[13] Alternatively, we could have modelled the commitment type as having the possibility of choosing either action, but with utilities such that U and R are dominant. In this case we would also need a refinement akin to the Intuitive Criterion of Cho and Kreps (1987), since merely using the concept of Sequential Equilibrium would not rule out "strange" beliefs on and off the equilibrium path.

[14] A different interpretation is that the time horizon is fixed, and the revision rates approach infinity while the ratio between them is preserved.

Definition 1. *A vector of parameters* $(u_1, u_2; \xi_1, \xi_2)$ ***induces***

- ***substantial delay*** *if there exists a time* $-t' < 0$ *and* $\delta > 0$*, such that for every sequence* $\{T^k\}_{k=1}^{\infty}$ *such that* $T^k \to \infty$*, and every corresponding sequence of SEs, the probability that only* (U, R) *is being prepared before time* $-t'$ *is bounded above* δ *as* k *approaches infinity.*

- ***last-minute strategic interaction*** *if there exists a time* $-t' < 0$ *and* $\delta > 0$*, such that for every sequence* $\{T^k\}_{k=1}^{\infty}$ *such that* $T^k \to \infty$*, and every corresponding sequence of SEs, the probability that the prepared profile changes between time* $-t'$ *and time* 0 *is bounded above* δ *as* k *approaches infinity.*

- ***inefficiency*** *if there exists* $\delta > 0$*, such that for every sequence* $\{T^k\}_{k=1}^{\infty}$ *such that* $T^k \to \infty$*, and every corresponding sequence of SEs, the probability of reaching ex-post inefficient payoffs is bounded above* δ *as* k *approaches infinity.*

Note that whenever a vector of parameters induces substantial delay, it also induces last-minute strategic interaction. When a vector of parameters induces last-minute strategic interaction it necessarily induces inefficiency (this is Lemma A.3 in the appendix).

2 One-sided reputation-building: Last-minute strategic interaction

We begin our analysis by focusing on one-sided reputation-building and its consequences. This simpler environment allows us to introduce some of the intuitions that continue to guide us when studying two-sided reputation-formation. The analysis is required for the study of two-sided reputation-formation also because it represents what happens after either of the players is revealed as a rational type. In what follows we establish first that even when the time horizon becomes long, a prior positive probability of acting "crazy" prevents an opponent from getting the entire surplus.[15]

To do this, we first rephrase and explain a useful definition of Calcagno *et al.* (2014) that summarizes several aspects of the players' bargaining power in our setting.

[15] As is the case in the complete information game (Calcagno *et al.*, 2014, Theorem 3).

Definition 2. *Player i's* ***strength*** *is given by*

$$s_i(u_i) \equiv \frac{|u_i(U,L) - u_i(D,R)|}{[u_i(U,L) - u_i(U,R)] + [u_i(D,R) - u_i(U,R)]}.$$

Player i is ***stronger*** *than Player j if $s_i(u_i) > s_j(u_j)$. Player i's* ***relative strength*** *(with regard to Player j) is*

$$\Delta_{ij}(u_1, u_2) = s_i(u_i) - s_j(u_j).$$

The denominator of the expression in Definition 2 can be thought of as the expected surplus of Player i when both players try to myopically best-respond over a time period, and there is an equal probability that any one of them will manage to stop before the deadline. This value is normalized by the difference between Player i's preferred component game equilibrium payoff and her less-preferred component game equilibrium payoff.[16] Player i is less strong if her preferred component game equilibrium payoff is higher, and is stronger if her less-preferred component game equilibrium payoff is higher, or if $u_i(U,R)$ is higher. Intuitively, lower payoffs for reaching the preferred outcome, and higher payoffs for reaching the less-preferred outcome and or for no agreement, all allow a player to insist longer. Note that for generic payoffs, $\Delta_{ij}(u_1, u_2) \neq 0$ and so one of the players is stronger than the other.[17]

Proposition 1. *Assume Player 1 is stronger than Player 2, $\xi_1 = 0$, and $\xi_2 > 0$; then the revision equilibrium payoff set is bounded away from Player 1's preferred outcome:*

$$u(U,L) \notin \bar{\phi}(u_1, u_2; 0, \xi_2).^{18}$$

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[16] Calcagno *et al.* (2014) note that there is an equivalent and perhaps more readable form for the inverse of s_i:

$$\frac{1}{s_i(u_i)} = 1 + 2 \cdot \frac{\min\{u_i(U,L), u_i(D,R)\} - u_i(U,R)}{|u_i(U,L) - u_i(D,R)|}.$$

[17] One may be tempted to compare this strength notion with the concept of risk-dominance (Harsanyi and Selten, 1988). Generally, these two measures do not agree. Note that our strength notion does not depend on $u(D,L)$, whereas the risk-dominance definition does take $u(D,L)$ into account.

[18] We write $u(U,L)$ instead of $(u_1(U,L), u_2(U,L))$, and similarly for all action pairs.

Proof sketch. The theorem shows that one-sided reputation-building bounds the revision equilibrium payoff set away from the stronger player's preferred outcome as the length of the time horizon tends to infinity. Intuitively, to get this outcome in the limit the rational Player 2 must prepare L on average further and further away from $t = 0$. But this implies that if the stronger player is called to prepare an action at any time before $t = 0$ and the weaker player's prepared action is the commitment action, then the stronger player will attribute a very high probability to the event that the weaker player is a commitment type, and will have a strict incentive to myopically best-respond from there on. This in turn creates an incentive for the rational type of the weaker player to imitate the commitment type, and we get a contradiction. The complete proof is in the appendix. □

To fully understand the role of the random processes in the derivation of the above result, it may be helpful to consider briefly a model in which the times of players' preparations are well known in advance. If there is no incomplete information about players' types, it is easy to see (using backward induction) that the last mover will have to prepare the action related to his less-preferred component game equilibrium. The same pair of strategies will also form a SE in an incomplete information game, as long as the probability of the other player being a commitment type is small enough. Generally, a deterministic order of play enhances such bargaining strengths as first-mover or last-mover advantage or disadvantage. When preparations are random both players can find themselves in a situation where they are the last player to prepare an action, which makes the exact order of preparations irrelevant in determining the bargaining strengths of the two players.

Proposition 1 indicates what kind of outcomes are impossible, but does not provide a full description of what does happen. For example, the expected payoffs may be bounded away from $u(U, L)$ due to some plays ending with Pareto inefficient outcomes, or the equilibrium behavior may dictate arriving at the weaker player's preferred outcome with a positive probability. Theorem 2 demonstrates that under certain parameters the equilibrium behavior necessarily involves last-minute strategic interaction that may lead to inefficient outcomes.

Theorem 2. *Assume Player 1 is stronger than Player 2, and $\xi_1 = 0$. Then there exists $\bar{\xi}_2 > 0$ such that for every $\xi_2 \in \left(0, \bar{\xi}_2\right)$, the parameters induce last-minute strategic interaction and inefficiency.*

Proof sketch. If the probability that Player 2 is a commitment type is small enough, then Player 1 can get a payoff strictly above $u_1(D, R)$ with high probability by never changing her action and waiting for Player 2 to best-respond. That is, some histories lead to Player 1 getting her most-preferred payoff, and Player 2 getting less than her most-preferred payoff. We know from Proposition 1 that Player 2 cannot prepare the action L too soon, and when she does not, Player 1 cannot always prepare D or else Player 2 will have a profitable deviation. This necessarily implies last-minute strategic interaction, which in turn directly causes inefficiency. The complete proof is in the appendix. □

We are inclined to interpret Theorem 2 as relating to different notions of advantage. These two flavors of strategic advantage are reminiscent of those discussed in the context of bargaining. For example, Rubinstein (1982) uses slight differences in impatience to determine the bargaining outcome, whereas Abreu and Gul (2000) suggest irrationality as an explanation for the final division of residual claims in the bargaining procedure. In our case the strength of Player 1 cannot prevent the rational type of Player 2 from imitating the commitment type and creating reputation. At the same time Player 1 cannot completely give up her strong bargaining posture too soon, as she knows that rationality dictates a change in the revisions of the rational type of Player 2 close to the deadline. As the probability of actually facing a commitment type becomes small, Player 1 has an incentive to preserve some bargaining power to the end of the revision phase. This means that for an open set of parameters, one-sided reputation-building leads to last-minute strategic interaction and inefficiency.[19] In Section 4 we strengthen this result for the case of War of Attrition with Poisson arrivals (Corollary 9).

An important question is whether the effect of reputation vanishes as the probability of a commitment type tends to zero. We show that in the case of unequal strengths indeed it does. We defer the discussion about the case of equal strengths to Section 3, where the case of two-sided reputation formation is studied. Here we bring a one-sided version of Theorem 6 that shows that any sequence of outcomes in games with diminishing probability for a commitment type tends to the outcome related to the subgame perfect equilibrium of the complete information game.

[19] Strictly speaking, we have not shown that the set of parameters is open. However, looking at the proof of Theorem 2, we can see that $\bar{\xi}_2$ can be chosen as a continuous function u_1 and u_2, and hence finding an open set of parameters is trivial.

Proposition 3. *Assume Player 1 is stronger than Player 2, and $\xi_1 = 0$. Then Player 1's preferred outcome is the only payoff in the limit of the revision equilibrium payoff set as $\xi_2 \to 0$:*[20]

$$\liminf_{\xi_2 \to 0} \bar{\phi}(u_1, u_2; 0, \xi_2) = \{u(U, L)\}$$

3 Two-sided reputation-building: Falling over the cliff

As demonstrated in the previous section, one-sided reputation formation may lead to inefficiency under certain conditions. The inefficiency is realized on the equilibrium path when the Democrats insist on keeping taxes high (U) and the Republicans insist on reducing spending (R), a scenario that leads to a fiscal cliff. The basic argument was that the Republicans, who are perceived as having a positive probability of being committed to reducing spending, cannot fold too quickly, or else the entire surplus will be taken by the Democrats who are stronger along other bargaining dimensions. This section applies a similar logic to situations with two-sided reputation-formation, and describes how it necessarily induces a positive probability of falling over the fiscal cliff. That is, the probability of seeing any action prepared in the game other than the commitment actions is bounded below 1 until the very end of the game, and the expected payoffs cannot be on the Pareto frontier. We characterize the inefficiency for the special case of equilibrium in cut-off strategies, and present comparative statics and limit results. Finally, we present a variety of numerical and analytic calculations that indicate that the inefficiency can be substantial for reasonable values of the parameters.

As described above, not only does our first theorem in this section predict a shift from the complete information outcome, but it also demonstrates that with positive probability a long delay must occur before any player prepares anything but the commitment action. This delay in turn must cause inefficiency, as agreement might not be reached before the deadline. Theorem 4 therefore combines and strengthens the results of Proposition 1 and Theorem 2 for the case of two-sided reputation-building.

[20] Note that $\bar{\phi}(u_1, u_2; 0, \xi_2)$ is a set. The operator lim inf here means that for every sequence $\xi_2^k \to 0$ and equilibrium payoffs $u^k \in \bar{\phi}(u_1, u_2; 0, \xi_2^k)$ we have $u^k \to u(U, L)$.

Theorem 4. *Assume Player* 1 *is stronger than Player* 2,[21] *and* $\xi_1, \xi_2 > 0$. *Then substantial delay and inefficiency are induced.*

The intuition for Theorem 4 is an extension of the one used to derive the one-sided reputation results. The rational type of the weaker player (in terms of payoffs) cannot give up on building reputation too quickly in equilibrium. If she does, then, by extension of Calcagno *et al.* (2014, Theorem 3), her payoff will approach her least preferred component game equilibrium payoff, while pretending to be a commitment type will guarantee a larger payoff. Given that there is a time before which the weaker player's probability of revealing her rationality is bounded below 1, there is also an earlier time before which the rational type of the stronger player cannot reveal her rationality too early either. If she does that her utility will be bounded away from her preferred outcome (by Proposition 1), and deviating ensures a payoff that approaches the preferred outcome. This dictates substantial delay, which necessarily leads to inefficiency.

The above theorem points to some of the problematic efficiency properties of two-sided reputation-building. The rest of this section is dedicated to uncovering the magnitude of the inefficiency, and how different parameters of the game affect it. In what follows we present some basic comparative statics regarding the extent of the inefficiency that Theorem 4 predicts, and show that for generic parameters the inefficiency disappears as the probability players being a commitment type go to zero. For some of our results we focus on the simple equilibrium form in which each rational type insists on her commitment action until some point in time and myopically best-responds (according to the component game's payoffs) from there on. We know that in order for these strategies to form an equilibrium, each rational type must be indifferent between both actions at the switching point, and this together with the Bayesian updating of beliefs allows us to solve for the cutoff times.

Definition 3. *Player i is said to be* ***playing a cutoff strategy*** *if her strategy for the rational type can be described as by one parameter $-t_i^*$, such that*

- *if called to prepare an action prior to $-t_i^*$, Player i prepares her commitment action (either U or R), and*

[21] This assumption is only for the sake of simplicity. The case of equal strengths is susceptible to similar analysis but requires proving first that if players are equally strong, $\xi_1 > 0$ and $\xi_2 = 0$, then equilibrium payoffs approach $u(U, L)$ as T approaches infinity.

- *if called to prepare an action at $-t_i^*$ or after it, Player i best-responds to the opponent's currently prepared action.*

We refer to $-t_i^*$ as Player i's ***cutoff****.*

Proposition 5. *If $\xi_1, \xi_2 > 0$, both players play cutoff strategies, and Player 2's cutoff is earlier than Player 1's cutoff, then the probability of reaching an ex-post Pareto inferior outcome is*

1. *increasing in $u_1(D,R)$, and decreasing in $u_1(U,L)$ and $u_1(U,R)$,*

2. *increasing in $u_2(D,R)$ and $u_2(U,R)$, and decreasing in $u_2(U,L)$.*

It is instructive to read Proposition 5 in light of Definition 2, and see that each element of the bargaining strength affects the inefficiency in the same way it affects the relative strength, $\Delta_{12}(u_1, u_2)$. Changes in the payoffs that push the relative strength away from zero decrease the inefficiency, and changes that push it towards zero increase the inefficiency. From symmetry it follows that the maximal inefficiency is attained when $\Delta_{12}(u_1, u_2) = 0$. To get a feeling of what that maximal inefficiency might be, Theorem 6 below takes both probabilities of being a commitment type to zero.

Theorem 6. 1. *Assume Player 1 is stronger than Player 2, and both players play cutoff strategies. Then in the limit as $\xi_1 \to 0$ and $\xi_2 \to 0$, the probability of reaching ex-post inefficiency tends to zero.*

2. *Assume players are equally strong ($\Delta_{12}(u_1, u_2) = 0$), and both players play cutoff strategies. Let the sequence $(\xi_1^k, \xi_2^k)_{k=1}^{\infty}$ be such that $\lim_{k\to\infty} \xi_1^k = \lim_{k\to\infty} \xi_2^k = 0$, and $\lim_{k\to\infty} \frac{\xi_2^k}{\xi_1^k} \leq 1$. Then in the limit as $k \to \infty$, the probability of reaching ex-post inefficiency tends to*

$$\left[\frac{u_1(U,L) - u_1(D,R)}{u_1(U,L) + u_1(D,R) - 2u_1(U,R)}\right] \times \lim_{k\to\infty} \frac{\xi_2^k}{\xi_1^k}.$$

The interpretation of Theorem 6 is that as the probability of commitment types goes to zero, the inefficiency (generically) goes to zero as well. Nevertheless, if the probability that either of the sides is a commitment type does not merely serve as a refinement tool, but rather constitutes an essential part of the dynamics, then the balance of the different forces becomes important. As the second part of the theorem shows, even for relatively small probabilities of being commitment types, the inefficiency can be substantial. Roughly, if the players are close to being equally strong in terms of payoffs, then reputation effect may cause significant efficiency loss. However, if one of the players is much stronger (in terms of payoffs), reputation effects only mildly hurt efficiency. Similarly, if one player's reputation is much stronger than the other player's (the ratio $\frac{\xi_2}{\xi_1}$ is very small or very big), then the inefficiency is negligible. Another way to understand this result is to think about the two parts of Theorem 6 as two different ways of taking limits. In the first part of the theorem we take commitment probabilities to zero (and then $\Delta_1 2(u_1, u_2)$ can be taken to zero as well), whereas the second part we first take $\Delta_1 2(u_1, u_2)$ to zero, and only then take the commitment probabilities to zero as well.

To illustrate the extent to which the equilibrium result may be inefficient we solve for a variety of parameters. We first note that when the two players are equally strong we can use the formula of Theorem 6 to compute the limit inefficiency. We assume $\xi_1 = \xi_2$ and (without loss of generality) normalize $u_1(U, R)$ to be zero, and get the reduced formula

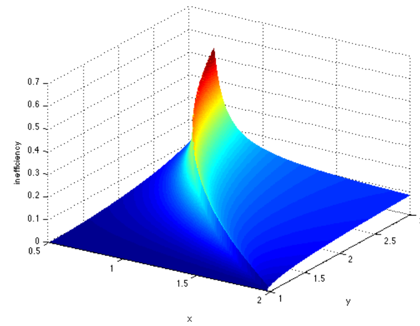
$$\text{Inefficiency} = \frac{u_1(U, L) - u_1(D, R)}{u_1(U, L) + u_1(D, R)}.$$

This implies that for a standard battle of the sexes payoff matrix, the inefficiency tends to $\frac{1}{3}$ in the limit. If the stakes are higher, say if $u_1(U, L) = u_2(D, R) = 99$ and $u_1(D, R) = u_2(U, L) = 1$, then the inefficiency is 98%. In this case equilibrium behavior dictates that both players insist on playing the commitment action until very close to time 0, and the first to be called to prepare an action chooses her less-preferred component game equilibrium and gets a payoff of 1, whereas the other player gets 99. However, the time at which players stop preparing the commitment action is so close to the deadline that only with probability 2% is either of them is going to be called, which pins their expected payoff at exactly 1.

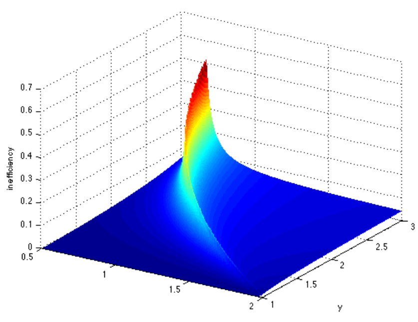
Figure 2 illustrates several of the properties that were just discussed. In this figure we see the extent of inefficiency for the game form presented in the top left panel for $x \in (0.5, 2)$ and $y \in (1, 3)$, and for three values of ξ_1 and ξ_2. The three shapes resemble shark fins that become narrower as ξ_1 and ξ_2 become smaller. This demonstrates that the maximal inefficiency is always at $\Delta_{12}(u_1, u_2) = 0$ and does not vanish, and that everywhere else the inefficiency tends to zero as the ex-ante probability of commitment types approaches zero. Figure 3 offers a different view of the same family of payoff

	L	R
U	$2, 1$	$0, 0$
D	$0, 0$	x, y

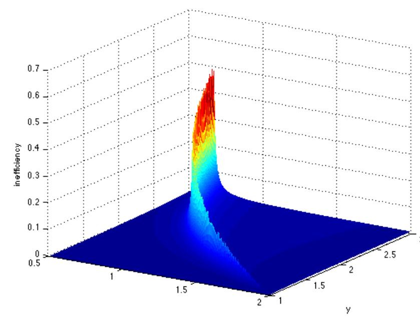
The payoff matrix



$\xi_1 = \xi_2 = 0.1$



$\xi_1 = \xi_2 = 0.05$



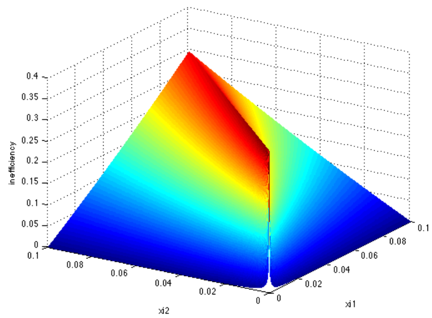
$\xi_1 = \xi_2 = 0.01$

Figure 2: Varying payoffs

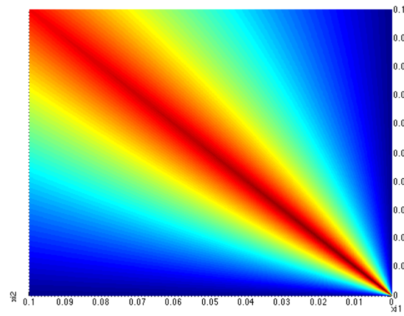
matrices, but here the payoffs are held constant and the ex-ante probabilities of commitment types vary. A careful look at those graphs reveals that there is a trade-off between strength (in terms of payoffs) and commitment power when it comes to bargaining posture, which in turn leads to inefficiency.

4 Wars of attrition with Poisson arrivals

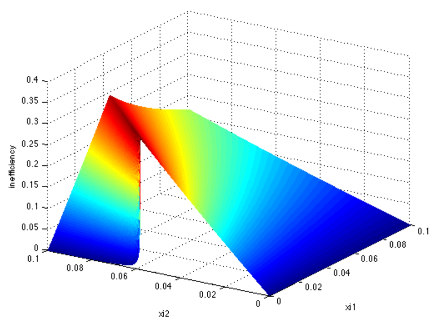
In many negotiations (including those related to the fiscal cliff), reaching a final agreement before the deadline is also a possibility. This is inherently different from the model of revision games in which even after one of the parties decides to concede, it can still return to its previous bargaining position. Assuming that a concession ends the game simplifies the model and



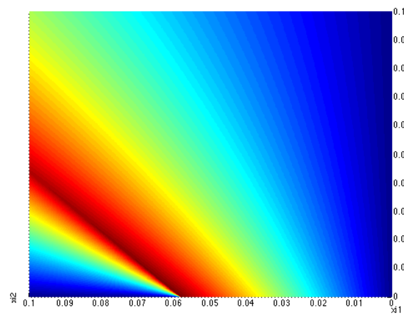
$x = 1\ ,\ y = 2$ (side)



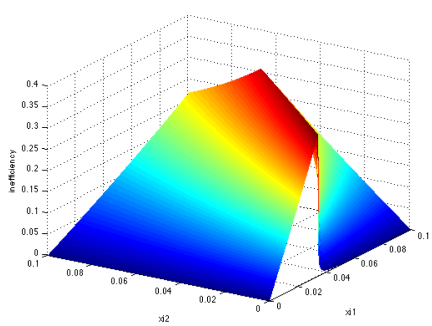
$x = 1\ ,\ y = 2$ (top)



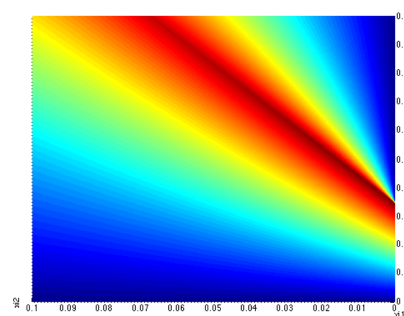
$x = 1\ ,\ y = 1.9$ (side)



$x = 1\ ,\ y = 1.9$ (top)



$x = 0.97\ ,\ y = 2$ (side)



$x = 0.97\ ,\ y = 2$ (top)

Figure 3: Varying commitment probabilities

turns it into a model of a war of attrition. Wars of attrition can generally be described as situations in which the first player to take action over some defined period of time determines the payoffs for both players. This model was first introduced by Maynard Smith (1974) to model biological situations in which two animals fight over a disputed territory. While the first animal to leave the territory receives a lower payoff than its rival, both animals suffer from the preceding fight. The framework also captures some of the economic incentives that govern firms when they are engaged in a patent race or when trying to become a monopolist in a market that cannot sustain multiple competitors. An extensive literature studies different aspects of the game.[22]

As mentioned in the Introduction, variants of wars of attrition that include reputation have been analyzed in the literature in both discrete and continuous-time. Here we introduce a new timing structure, namely, continuous and finite time, where players can exit only when being called to play by a stochastic process. We restrict ourselves to payoffs that correspond to a 2x2 opposing interests game when the stochastic processes governing the timing of the players' decisions are independent Poisson processes. One specific instance from this class of games is the "three-state example" analyzed (without reputation effects) by Kamada and Sugaya (2010, Section 6.1).

Roughly speaking, all of the previous results hold for this class of games. However, it turns out that in the war of attrition with Poisson arrivals all equilibria have the special structure of both players exiting with probability one when being called to play after some cutoff time, and at least one of the two players exiting with zero probability before that time. Furthermore, every equilibrium payoff can be mimicked using cutoff strategies for both players. We use these observations to pin down the limit results.

The fact that the strategies are simpler allows us to provide even sharper results. We continue to use the same basic definitions of types and utilities (see the appendix for formal definitions of histories, information sets, and strategies). As before, we now let $\phi^{\text{woa}}(T; u_1, u_2; \xi_1, \xi_2)$ denote the set of interim SE payoffs of the profile (τ_1^r, τ_2^r) in the war of attrition with Poisson arrivals and payoffs given by u_1 and u_2. We define $\bar{\phi}^{\text{woa}}(u_1, u_2; \xi_1, \xi_2)$ as the limit payoffs set of this game as T approaches infinity.

[22]Such as random rewards (Bishop *et al.*, 1978), asymmetric equilibria (Nalebuff and Riley, 1985), effects of continuous-time (Hendricks *et al.*, 1988), strong evolutionary equilibria (Riley, 1980), and more.

Lemma 7. *In the war of attrition with Poisson arrivals (and incomplete information), for any T, all sequential equilibria have the property that there exists a cutoff time $-t^*$ such that both players exit if being called to play after $-t^*$, and at least one of them exits with zero probability before $-t^*$.*

Corollary 8. *In the war of attrition with Poisson arrivals (and incomplete information), for any T all SE payoffs can be attained by SEs in which both players use cutoff strategies.*

The intuition for the proof of Lemma 7 is that players can use strategies that do not depend on their previous decision opportunities because the other player's inference does not depend on those opportunities. Furthermore, the nature of players' incentives is such that if one player exits at some time $-t$, and there is a strictly positive probability that the second wants to exit from then on until some future time $-t' > -t$, then after time $-t'$ the first player will definitely want to exit. This "pairwise monotonicity" property, which is precisely defined in the appendix, can be used to show that all equilibria have a cutoff time $-t^*$. Corollary 8 then follows by reducing the probability with which the "weaker" player exits before $-t^*$ to an interval just before $-t^*$, while preserving the conditions for equilibrium and giving the same expected payoffs as the original SE.

Lemma 7 and Corollary 8 represent the key difference between the analysis presented in the previous sections and the analysis of a war of attrition. Specifically, we can use Lemma 7 and Corollary 8 and repeat the same arguments from before to get sharper results. In the case of one-sided reputation formation, the results about efficiency and inefficiency under certain parameters are stronger, and equilibrium payoffs are uniquely determined for small enough values of ξ_2. Uniqueness is established by ruling out any other equilibrium payoffs using only cutoff strategies. Then the construction that appears in the proof of Proposition 3 nails down the exact values of the cutoff times. These results again demonstrate that the ability to build reputation is an extremely important property of the bargaining process.

Corollary 9. *In a war of attrition model, assume Player* 1 *is stronger than Player* 2 *and* $\xi_1 = 0$*. Then there exists* $\bar{\xi}_2$ *such that for every* $\xi_2 \in (0, \bar{\xi}_2)$*, the parameters induce substantial delay and inefficiency.*

Corollary 10. *In a war of attrition model, assume Player* 1 *is stronger than Player* 2*; then her preferred outcome is the unique limit of the equilibrium* *payoff set as the probability of Player* 2 *being the commitment type approaches zero. Formally,* $\liminf_{\xi_2 \to 0} \bar{\phi}^{woa}(u_1, y_2; 0, \xi_2) = \{u(U, L)\}$.

As for the case of two-sided reputation-building, all the results from Section 3 hold, but whenever the restriction about cutoff strategies appears it is redundant. We here avoid repeating these results or providing their proofs, which resemble the proofs that were used for the revision game model but also employ Corollary 8.

For the sake of completeness it is useful to review how our results relate to previous models of the war of attrition with incomplete information in continuous-time. First, as mentioned before, the continuous-time framework makes proving uniqueness results far more complicated than the results we present here. Our model relies on discrete actions, and so we can more easily apply the sequential rationality property to establish uniqueness. Second, a striking feature is that the rates at which players exit in the war of attrition are similar. Since we employ cutoff strategies, players exit according to the two Poisson processes that govern their action opportunities starting from a certain time. Osborne (1985) provides an expression that amounts to exactly the same exit rate when the players are risk-neutral.

5 Conclusion

While the fiscal cliff negotiations ended with an agreement between the two parties, similar bargaining situations often lead to an outcome undesirable to both sides in the form of passing the deadline. While ignoring some real-life reasons that may cause this to happen (animosity, miscommunication, and so on), we provide an intuitive explanation for such events. Our prediction of non-degenerate equilibrium play with the possibility of a deadline effect stands in sharp contrast with previous results on complete information play, and suggests that reputation formation may play a crucial part in delaying meaningful bargaining in the presence of a deadline.

It is likely that the effects mentioned in this paper would only become stronger if there were an element of repeated bargaining between the parties, as is natural in the political arena. In this case, reputation formation would not only serve the present agenda, but would also be beneficial for future negotiations. A different direction in which our results can be extended is to consider a game with incomplete information on the rates of revision opportunities. In such a game each player may try to persuade her opponent that her rate of revision opportunities is low. This creates an incentive for the opponent to give up her bargaining posture faster than she would have done had she known she was facing a player who can still react with high probability before the deadline. This particular kind of incomplete information may serve as a good model of the level of interest that each of the players invests in the game, and provides yet another reason why even players who care dearly about the results of the game would like to hold their position as the deadline approaches.

Finally, it is possible to consider an alternative model in which players' revision opportunities are observed by their opponents. While non-observable arrivals of revision opportunities are slightly more intuitive, the alternative model may also be relevant for certain applications. However, once players' revision opportunities become common knowledge delicate issues of playing mixed strategies arise. Studying the structure of strategies in a war of attrition with Poisson arrivals and incomplete information under this observability condition also becomes tricky, and it remains an open question whether or not cutoff strategies can be used to exhaust the entire space of equilibrium payoffs.

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A Appendix

A.1 Formal Definition of the Extensive-Form Game

Revision games: The set of players is $N = \{1, 2\}$ and the sets of types are $\mathcal{T}_i = \{\tau_i^r, \tau_i^c\}$. The set of possible histories is

$$\mathcal{H} = \{\emptyset\} \cup \left\{ -t, (\tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2, x) \;\middle|\; \begin{array}{l} \tau_i \in \mathcal{T}_i, \mathcal{O}_i \subseteq [-T, 0], -T \in \mathcal{O}_i, \\ x \colon [-T, -t) \to \{U, D\} \times \{L, R\} \end{array} \right\}.$$

Here $\mathcal{O}_i$ represents Player i's revision opportunities, and $x(-t')$ represents the prepared profile at time $-t'$. For almost all realizations $\mathcal{O}_1$ and $\mathcal{O}_2$ will be finite. The players who take action at h are given by the function $P \colon \mathcal{H} \to 2^{N \cup \{\text{Nature}\}}$:

$$P(h) = \begin{cases} \{\text{Nature}\} & \text{if } h = \emptyset \\ \{i \mid -t \in \mathcal{O}_i\} & \text{if } h = (-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2 x) \end{cases}.$$

The information sets partition for Player i is given by $\mathcal{I}_i$ whose typical element is

$$\mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i, \tilde{x}\right) =$$
$$\left\{ (-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2, x) \;\middle|\; -t = -\tilde{t}, \tau_i = \tilde{\tau}_i, \mathcal{O}_i|_{[-T,-t]} = \tilde{\mathcal{O}}_i, x|_{[-T,-t)} = \tilde{x} \right\}.$$

The set of information sets in which Player i takes an action is

$$\mathcal{J}_i = \left\{ \mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i, \tilde{x}\right) \in \mathcal{I}_i \;\middle|\; -\tilde{t} \in \tilde{\mathcal{O}}_i \right\}.$$

At history $\emptyset$ Nature chooses τ_1, τ_2, $\mathcal{O}_1$, and $\mathcal{O}_2$ independently according to the probabilities ξ_1 and ξ_2 (for τ_1^c and τ_2^c respectively) and according to the distributions of the Poisson processes with frequencies λ_1 and λ_2 (to determine $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively). The game then moves immediately to the history $(\tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2, -T, x^{\emptyset})$, with the "empty" function $x^{\emptyset}\colon \emptyset \to \{U, D\} \times \{L, R\}$. Following that, time moves continuously and whenever a player is called to play the available actions are

$$A_i\left(\mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i, \tilde{x}\right)\right) = \begin{cases} \{U, D\} & \text{if } i = 1, \tilde{\tau}_i = \tau_i^r \\ \{U\} & \text{if } i = 1, \tilde{\tau}_i = \tau_i^c \\ \{L, R\} & \text{if } i = 2, \tilde{\tau}_i = \tau_i^r \\ \{R\} & \text{if } i = 2, \tilde{\tau}_i = \tau_i^c \end{cases}$$

We denote also $A_1 = \{U, D\}$ and $A_2 = \{L, R\}$.

A feasible strategy for Player i is $\sigma_i\colon \mathcal{J}_i \to \Delta A_i$ such that

1. $\operatorname{supp}\left(\sigma_i\left(\mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i, \tilde{x}\right)\right)\right) \subseteq A_i\left(\mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i, \tilde{x}\right)\right)$
2. $\forall a_i \in A_i : \sigma_i\left(\mathcal{I}_i\left(-T, \tau_i^r, \{-T\}, x^{\emptyset}\right)\right)[a_i] \in \{0, 1\}$

As mentioned we restrict players' strategies to be measurable with respect to the natural topologies.

The state variable x is determined by the realizations of players' strategies. For any time $-t' < -t$, let $-t_i'' = \max\{-\tau \in \mathcal{O}_i \mid -\tau < -t\}$, and let α_i be the realized action of Player i at $-t_i''$, then $x(-t') = (\alpha_1, \alpha_2)$.

The terminal histories are $\mathcal{Z} = \{(-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2, x) \in \mathcal{H} \mid -t = 0\}$, and the payoffs at history $(0, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2, x)$ are given by $u\,(x(0))$, where u is a function from $\{U, D\} \times \{L, R\}$ to $\mathbb{R}^2$.

<u>Wars of attrition with Poisson arrivals:</u> In the case of the war of attrition with Poisson arrivals the definition of the game is slightly simpler.

The set of possible histories is now given by

$$\mathcal{H} = \{\emptyset, D, L\} \cup \{(-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2) \mid \tau_i \in \mathcal{T}_i, \mathcal{O}_i \subseteq [-T, 0], -T \in \mathcal{O}_i\}$$

The information sets partition for Player i is given by $\mathcal{I}_i$ whose typical element is

$$\mathcal{I}_i\left(-\tilde{t}, \tilde{\tau}_i, \tilde{\mathcal{O}}_i\right) = \left\{(-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2) \;\middle|\; -t = -\tilde{t}, \tau_i = \tilde{\tau}_i, \mathcal{O}_i|_{[-T,-t]} = \tilde{\mathcal{O}}_i\right\}.$$

The rest of the details are identical to those given for revision games, except that once Player 1 chooses action D or Player 2 chooses action L the game moves to the relevant history (D or L). The terminal histories are $\mathcal{Z} = \{D, L\} \cup \{(-t, \tau_1, \tau_2, \mathcal{O}_1, \mathcal{O}_2) \in \mathcal{H} \mid -t = 0\}$, and payoffs are given by

$$u^{\text{woa}}(h) = \begin{cases} u(D,R) & \text{if } h = D \\ u(U,L) & \text{if } h = L \\ u(U,R) & \text{otherwise} \end{cases}$$

with u defined as before.

A.2 Model with Heterogeneous Revision Rates

The main text ignored the possibility of the two players having different revision rates. Here we formalize all the necessary definitions to deal with heterogeneous revision rates. All the proofs below are provided for the more general case of different frequencies.[23] The game is now summarized by the parameters $(T; u_1, u_2; \xi_1, \xi_2; \lambda_1, \lambda_2)$, where λ_i is the frequency of the Poisson process governing Player i's revisions. We denote by $\phi(T; u_1, u_2; \xi_1, \xi_2; \lambda_1, \lambda_2)$ the set of interim SE payoffs of the profile (τ_1^r, τ_2^r), and define the revision equilibrium payoff set of $(u_1, u_2; \xi_1, \xi_2; \lambda_1, \lambda_2)$ by

$$\bar{\phi}(u_1, u_2; \xi_1, \xi_2, \lambda_1, \lambda_2) = \lim_{T' \to \infty} \phi(T'; u_1, u_2; \xi_1, \xi_2; \lambda_1, \lambda_2).$$

We extend Definition 1 trivially.

Lastly, the definition of strength becomes slightly more complicated.

Definition 2. *Player i's **strength** is given by*

$$s_i(u_i; \lambda_1, \lambda_2) \equiv \frac{\lambda_{3-i} \cdot |u_i(U,L) - u_i(D,R)|}{\lambda_2 \left[u_i(U,L) - u_i(U,R)\right] + \lambda_1 \left[u_i(D,R) - u_i(U,R)\right]}.$$

*Player i is **stronger** than Player j if*

$$s_i(u_i; \lambda_1, \lambda_2) > s_j(u_j; \lambda_1, \lambda_2).$$

*Player i **relative strength** (with regard to Player j) is*

$$\Delta_{ij}(u_1, u_2; \lambda_1, \lambda_2) = s_i(u_i; \lambda_1, \lambda_2) - s_j(u_j; \lambda_1, \lambda_2).$$

[23]With the exception of Theorem 6.

Note that a player is stronger if her revision frequency is low or if her opponent's revision frequency is high. The relative frequency of play allows a player to commit and thus makes her a stronger competitor (a player whose frequency of play is much lower than her opponent is practically a Stackelberg leader).

A.3 Proofs

Proposition 1. *Assume Player* 1 *is stronger than Player* 2*,* $\xi_1 = 0$*, and* $\xi_2 > 0$*; then the revision equilibrium payoff set is bounded away from Player* 1*'s preferred outcome:*

$$u(U,L) \notin \bar{\phi}(u_1, u_2; 0, \xi_2; \lambda_1, \lambda_2).$$

Proof. Let $\{T^k\}_{k=1}^{\infty}$ be a sequence of horizons such that $T^k \to \infty$ and let $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ be a corresponding sequence of equilibria. We let P^k denote the probability measure induced by Nature's moves (i.e., the lottery over types and the stochastic Poisson processes governing players' revision opportunities) and by equilibrium strategies $\hat{\sigma}^k$.

Choose $K^A > 0$, $-t^A < 0$ and $\delta^A > 0$ such that

1. $\delta^A < \frac{1}{3}\left(u_2(D,R)) - u_2(U,L)\right)$,
2. $e^{-\frac{1}{2}\lambda_1 t^A}\left(u_2(D,R) - u_2(U,R)\right) < \delta^A$,
3. for any $k > K^A$, given that it is common knowledge that Player 2 is rational by time $-t^A$, the expected continuation payoffs induced by equilibrium strategies $\hat{\sigma}^k$ are below $u_2(U,L) + \delta^A$ for Player 2.

The reason we can find K^A, $-t^A$, and δ^A that meet these conditions is due to Lemma A.1 below, the proof of which is a straightforward extension of the proof of Theorem 3 of Calcagno *et al.* (2014) and it is omitted.

Lemma A.1. *Assume Player* 1 *is stronger than Player* 2*, and* $\xi_2 = 0$*. Then the revision equilibrium payoff set contains only Player* 1*'s preferred outcome:*

$$\bar{\phi}(u_1, u_2; \xi_1, 0; \lambda_1, \lambda_2) = \{u(U,L)\}.$$

Assume to the contrary that $u(U, L) \in \bar{\phi}(u_1, u_2; 0, \xi_2; \lambda_1, \lambda_2)$. This implies that $\lim_{k\to\infty} P^k \left(E_L(-t^A) \mid E_{2r}\right) = 1$, where

$$
\begin{aligned}
E_{2r} &= \{\text{Player 2 is rational}\} \\
E_L(-t) &= \{L \text{ is prepared before time } -t\}
\end{aligned}
$$

(or else the equilibrium result will be bounded away from the intended limit). Then for any $\delta' > 0$ there exists $K' > K^A$ such that for any $k > K'$ Player 2's expected utility under $\hat{\sigma}^k$ can be bounded above by

$$(1 - \delta') \cdot \left(u_2(U, L) + \delta^A\right) + \delta' \cdot u_2(D, R) \leq u_2(U, L) + \delta^A + \delta' u_2(D, R). \quad (1)$$

We can take $\delta' = \frac{\delta^A}{u_2(D,R)}$ and get that there exists $K' > 0$ such that for any $k > K'$ Player 2's expected utility is bounded above by $u_2(U, L) + 2\delta^A$.

The rational Player 2 can deviate to the strategy in which she prepares R until time $-\frac{t^A}{2}$, and then best-responds to the prepared profile. Note that

$$\lim_{k\to\infty} P^k \left(E_{2r}^c \mid \left[E_L(-t^A)\right]^c\right) = \lim_{k\to\infty} \frac{\xi_2}{\xi_2 + (1 - \xi_2) \cdot P^k \left(\left[E_L(-t^A)\right]^c \mid E_{2r}\right)} = 1.$$

And therefore for large enough k Player 1's strategy from $-t^A$ onward will be to prepare D conditional on Player 2 never preparing L. This will ensure the rational Player 2 an expected utility bounded below by

$$\left(1 - e^{-\frac{1}{2}\lambda_1 t^A}\right) u_2(D, R) + e^{-\frac{1}{2}\lambda_1 t^A} u_2(U, R) > u_2(D, R) - \delta^A. \quad (2)$$

By the definition of δ^A, (1), and (2), it follows that Player 2 has a profitable deviation and we get a contradiction. □

Theorem 2. *Assume Player 1 is stronger than Player 2, and $\xi_1 = 0$. Then there exists $\bar{\xi}_2$ such that for every $\xi_2 \in \left(0, \bar{\xi}_2\right)$, the parameters induce last-minute strategic interaction and inefficiency.*

Proof. Let $\bar{\xi}_2$ be implicitly defined by

$$\frac{\bar{\xi}_2}{1 - \bar{\xi}_2} = \Delta_{12}(u_1, u_2; \lambda_1, \lambda_2) \cdot \frac{\lambda_1 u_1(D, R) + \lambda_2 u_1(U, L) - (\lambda_1 + \lambda_2) u_1(U, R)}{(\lambda_1 + \lambda_2) u_1(D, R)}.$$

The reasons for using this definition will become clear in the proof.

Given some $\xi_2 \in \left(0, \bar{\xi}_2\right)$, let $\{T^k\}_{k=1}^{\infty}$ be a sequence of horizons such that $T^k \to \infty$ and let $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ be a corresponding sequence of equilibria. We let P^k denote the probability measure induced by Nature's moves (i.e., the lottery over types and the stochastic Poisson processes governing players' revision opportunities) and by equilibrium strategies $\hat{\sigma}^k$.

Note that the rational type of Player 2 must myopically best-respond after time $-t_2^*$, where $-t_2^*$ is implicitly defined by

$$u_2(U,L) = \left(1 - e^{-(\lambda_1+\lambda_2)t_2^*}\right) \cdot \frac{\lambda_1 u_2(D,R) + \lambda_2 u_2(U,L)}{\lambda_1 + \lambda_2} + e^{-(\lambda_1+\lambda_2)t_2^*} u_2(U,R),$$

or more simply

$$e^{-(\lambda_1+\lambda_2)t_2^*} = s_2(u_2; \lambda_1, \lambda_2).$$

Let $-t^B$ be such that $-t^B \leq \min\{-t^A, -t_2^*\}$, with $-t^A$ being defined as in Proposition 1. Using the notation defined in the proof of Proposition 1 we know that $\lim_{k\to\infty} P^k\left(E_L(-t^B) \mid E_{2r}\right) < 1$.[24]

Assume to the contrary that there is no last-minute strategic interaction. This implies that the probability of the profile of prepared actions changing in the interval $(-t^B, 0)$ goes to zero as k goes to infinity. Let $E_{(a_1,a_2)}$ denote the event that (a_1, a_2) is the prepared action profile at time $-t^B$.

Claim 2.1. $\lim_{k\to\infty} P^k(E_{(U,R)} \mid E_{2r}) = \lim_{k\to\infty} P^k(E_{(D,L)} \mid E_{2r}) = 0.$

Proof. If the action profile prepared at time $-t^B$ is either (U,R) or (D,L) there is a constant probability that Player 2 will be called to play after time $-t_2^*$, and will best-respond to the prepared action of Player 1, which contradicts the lack of last-minute strategic interaction. ■

Claim 2.2. $\lim_{k\to\infty} P^k(E_{(U,L)} \mid E_{2r}) > 0.$

Proof. If $\lim_{k\to\infty} P^k(E_{(U,L)} \mid E_{2r}) = 0$ then for any $\delta > 0$, Player 1's utility for large enough k is bounded above by $u_1(D,R) + \delta$. However, Player 1 can deviate to the strategy of playing only U until time $-t_2^*$ and then best-

[24] For convenience, and without loss of generality, we will assume throughout the proof that all the sequences of probabilities we mention converge.

responding, so that she will get a payoff bounded below by

$$
\begin{aligned}
&(1-\xi_2)\cdot\Big(\left(1-e^{-(\lambda_1+\lambda_2)t_2^*)}\right)\cdot\frac{\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)}{\lambda_1+\lambda_2}+\\
&e^{-(\lambda_1+\lambda_2)t_2^*}u_1(U,R)\Big)+\xi_2\cdot\left(\left(1-e^{-\lambda_1 t_2^*}\right)u_1(D,R)+e^{-\lambda_1 t_2^*}u_1(U,R)\right)>\\
&\frac{1-\xi_2}{\lambda_1+\lambda_2}\cdot\Big(\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)-s_2(u_2;\lambda_1,\lambda_2)\cdot\\
&(\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)-(\lambda_1+\lambda_2)u_1(U,R))\Big)=\\
&\frac{1-\xi_2}{\lambda_1+\lambda_2}\cdot\Big(\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)+(\Delta_{12}(u_1,u_2;\lambda_1,\lambda_2)-s_1(u_1;\lambda_1,\lambda_2))\cdot\\
&(\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)-(\lambda_1+\lambda_2)u_1(U,R))\Big)=(1-\xi_2)u_1(D,R)+\\
&(1-\xi_2)\Delta_{12}(u_1,u_2;\lambda_1,\lambda_2)\cdot\frac{\lambda_1 u_1(D,R)+\lambda_2 u_1(U,L)-(\lambda_1+\lambda_2)u_1(U,R)}{\lambda_1+\lambda_2}>\\
&(1-\xi_2)u_1(D,R)+\xi_2 u_1(D,R)=u_1(D,R),
\end{aligned}
$$

where the last inequality comes from ξ_2 being strictly less than $\bar{\xi}_2$ and the definition of $\bar{\xi}_2$. The fact that we have a lower bound that is strictly above $u_1(D,R)$ implies that we can select δ small enough such that the deviation will be profitable for Player 1 for large enough k. ■

Claim 2.3. $\lim_{k\to\infty} P^k\left(E_{(D,R)} \mid \left[E_L(-t^B)\right]^c \cap E_{2r}\right) < 1$.

Proof. Suppose $\lim_{k\to\infty} P^k\left(E_{(D,R)} \mid \left[E_L(-t^B)\right]^c \cap E_{2r}\right) = 1$. We know from Claim 2.2 that for every $\delta' > 0$ there exists K' such that for every $k > K'$ Player 2's payoffs from equilibrium $\hat{\sigma}^k$ are bounded from above by

$$u_2(D,R) - \left(\lim_{k\to\infty} P^k\left(E_{(U,L)} \mid E_{2r}\right) - \delta'\right)\cdot\left(u_2(D,R) - u_2(U,L)\right).$$

Player 2 can deviate and always prepare R and for any $\delta'' > 0$ there exists K'' such that for any $k > K''$ Player 2's payoffs from this deviation will be bounded from below by

$$u_2(D,R) - \delta''\cdot u_2(U,R). \tag{3}$$

If we pick δ' and δ'' to be small enough, and take $K = \max\{K', K''\}$, Player 2 has a profitable deviation for all $k > K$. ■

From Claim 2.3 and Proposition 1 we know that

$$\lim_{k\to\infty} P^k\left(E_{(U,R)} \mid E_{2r}\right) = \lim_{k\to\infty} P^k\left(E_{(U,R)} \mid \left[E_L(-t^B)\right]^c \cap E_{2r}\right) \cdot \lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_{2r}\right) > 0,$$

which contradicts Claim 2.1. This concludes the contradiction argument and proves that the parameters induce last-minute strategic interaction. The proof is completed using Lemma A.3 below. □

Lemma A.3. *If parameter vector $(u_1, u_2; \xi_1, \xi_2; \lambda_1, \lambda_2)$ induces last-minute strategic interaction, then it induces inefficiency.*

Proof. Given the vector of parameters, we know that there exists $-t' < 0$ and $\delta > 0$ such that for every sequence $\{T^k\}_{k=1}^{\infty}$ such that $T^k \to \infty$, and every corresponding sequence of SEs, the probability that the prepared profile changes between time $-t'$ and time 0 is bounded above δ as k approaches infinity. Note that two of the four possible action profiles are inefficient and there is no way to change from one efficient profile to the other without passing through an inefficient profile. Therefore, there is a probability of at least δ of reaching an inefficient profile at some time between $-t'$ and 0. It is therefore possible to bound the inefficiency from below by δ times the probability that no player moves from the time an inefficient profile was reached until 0, that is, $\delta' = \delta \cdot e^{-(\lambda_1+\lambda_2)t'}$. □

Proposition 3. *Assume Player 1 is stronger than Player 2, and $\xi_1 = 0$. Then Player 1's preferred outcome is the only payoff in the limit of the revision equilibrium payoff set as $\xi_2 \to 0$:*

$$\liminf_{\xi_2\to 0} \bar{\phi}(u_1, u_2; 0, \xi_2; \lambda_1, \lambda_2) = \{u(U, L)\}$$

Proof. Proof is omitted (in process of revision). □

Theorem 4. *Assume Player 1 is stronger than Player 2, and $\xi_1, \xi_2 > 0$. Then substantial delay and inefficiency are induced.*

Proof. We will first show that substantial delay is induced, and this in turn will imply last-minute strategic interaction and inefficiency (via Lemma A.3).

Let $\{T^k\}_{k=1}^{\infty}$ be a sequence of horizons such that $T^k \to \infty$ and let $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ be a corresponding sequence of equilibria. We let P^k denote the probability measure induced by Nature's moves (i.e., the lottery over types and the stochastic Poisson processes governing players' revision opportunities) and by equilibrium strategies $\hat{\sigma}^k$.

Choose $K^A > 0$, $-t^A < 0$ and $\delta^A > 0$ such that

1. $\delta^A < \frac{1}{3}(1-\xi_1)\left(u_2(D,R)) - u_2(U,L)\right)$,
2. $e^{-\frac{1}{2}\lambda_1 t^A}\left(u_2(D,R) - u_2(U,R)\right) < \delta^A$,
3. $e^{-\frac{1}{2}\lambda_2 t^A}\left(u_2(U,L) - u_2(U,R)\right) < \delta^A$,
4. for any $k > K^A$, given that it is common knowledge that Player 2 is rational by time $-t^A$, the expected continuation payoffs induced by equilibrium strategies $\hat{\sigma}^k$ are above $u_1(U,L) - \delta^A$ for Player 1 and below $u_2(U,L) + \delta^A$ for Player 2.

The reason we can find K^A, $-t^A$, and δ^A that meet these conditions is due to Lemma A.1.

We describe several events using the following notation:

$$\begin{aligned}
&E_{1r} = \{\text{Player 1 is rational}\}\\
&E_{2r} = \{\text{Player 2 is rational}\}\\
&E_r = E_{1r} \cap E_{2r}\\
&E_D(-t) = \{D \text{ is prepared before time } -t\}\\
&E_L(-t) = \{L \text{ is prepared before time } -t\}
\end{aligned}$$

Assume to the contrary that no substantial delay is induced. This implies that the probability that by time $-t^A$ the only profile that was prepared was (U,R) conditional on both players being rational converges to zero as $k \to \infty$. Formally:

$$\lim_{k\to\infty} P^k\left(\left[E_D(-t^A)\right]^c \cap \left[E_L(-t^A)\right]^c \mid E_r\right) = 0.^{25} \tag{4}$$

Claim 4.1. $\lim_{k\to\infty} P^k\left(\left[E_L(-t^A)\right]^c \mid E_r\right) > 0.$

[25] For convenience, and without loss of generality, we will assume throughout the proof that all the sequences of probabilities we mention converge.

Proof. Suppose that $\lim_{k\to\infty} P^k\left(\left[E_L(-t^A)\right]^c \mid E_r\right) = 0$; then for any $\delta' > 0$ there exists $K' > K^A$ such that for any $k > K'$ Player 2's expected utility under $\hat{\sigma}^k$ can be bounded above by

$$\begin{aligned}&\xi_1 \cdot u_2(U,L) + (1-\xi_1)\cdot\left((1-\delta')\cdot\left(u_2(U,L)+\delta^A\right) + \delta'\cdot u_2(D,R)\right) \\ &\leq u_2(U,L) + \delta^A + \delta' u_2(D,R).\end{aligned} \tag{5}$$

We can take $\delta' = \frac{\delta^A}{u_2(D,R)}$ and get that there exists $K' > 0$ such that for any $k > K'$ Player 2's expected utility is bounded above by $u_2(U,L) + 2\delta^A$.

The rational Player 2 can deviate to the strategy in which she prepares R until time $-\frac{t^A}{2}$, and then best-responds to the prepared profile. Note that

$$\begin{aligned}&\lim_{k\to\infty} P^k\left(E_{2r}^c \mid \left[E_L(-t^A)\right]^c \cap E_{1r}\right) = \\ &\lim_{k\to\infty} \frac{\xi_2}{\xi_2 + (1-\xi_2)\cdot P^k\left(\left[E_L(-t^A)\right]^c \mid E_r\right)} = 1.\end{aligned}$$

And therefore for large enough k the rational Player 1's strategy from $-t^A$ onward will be to prepare D conditional on Player 2 never preparing L. This will ensure the rational Player 2 an expected utility bounded below by

$$\begin{aligned}&\xi_1\cdot\left(\left(1-e^{-\frac{1}{2}\lambda_2 t^A}\right)u_2(U,L) + e^{-\frac{1}{2}\lambda_2 t^A}u_2(U,R)\right) + \\ &(1-\xi_1)\cdot\left(\left(1-e^{-\frac{1}{2}\lambda_1 t^A}\right)u_2(D,R) + e^{-\frac{1}{2}\lambda_1 t^A}u_2(U,R)\right) > \\ &\xi_1\cdot u_2(U,L) + (1-\xi_1)\cdot u_2(D,R) - \delta^A.\end{aligned} \tag{6}$$

From the definition of δ^A, (5), and (6), it follows that Player 2 has a profitable deviation, and we get a contradiction. ■

Following Claim 4.1, let $\bar{p} \equiv \lim_{k\to\infty} P^k\left(\left[E_L(-t^A)\right]^c \mid E_r\right)$. Choose $K^B > K^A$, $-t^B \leq -t^A$ and $\delta^B > 0$ such that

1. for any $k > K^B$, given that it is common knowledge by time $-t \leq -t^B$ that Player 1 is rational, and that Player 1 believes that Player 2 is a commitment type with probability at least ξ_2, the expected continuation payoffs induced by equilibrium strategies $\hat{\sigma}^k$ are below $u_1(U,L) - \delta^B$ for Player 1,

2. $e^{-\frac{1}{2}\lambda_1 t^B}\left(u_1(U,L) - u_1(U,R)\right) < \frac{1}{2}(1-\xi_2)\bar{p}\delta^B$.

We can select such K^B, t^B, and δ^B as shown by Theorem 2 (note that the method of proof there implied that payoffs can be bounded for all times before $-t^B$ simultaneously).

Claim 4.2. $\lim_{k\to\infty} P^k\left(E_D(-t^B) \mid \left[E_L(-t^B)\right]^c \cap E_r\right) < 1$.

Proof. Note that Claim 4.1 and (4) imply that

$$\lim_{k\to\infty} P^k\left(E_D(-t^B) \cap \left[E_L(-t^B)\right]^c \mid E_r\right) > 0.$$

Assume to the contrary that $\lim_{k\to\infty} P^k\left(E_D(-t^B) \mid \left[E_L(-t^B)\right]^c \cap E_r\right) = 1$. Then

$$\lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_r\right) = \lim_{k\to\infty} P^k\left(E_D(-t^B) \cap \left[E_L(-t^B)\right]^c \mid E_r\right) > 0.$$

This in turn implies that for any $\delta' > 0$ there exists $K' > K^B$ such that the rational type of Player 1's expected utility under $\hat{\sigma}^k$ can be bounded above by

$$\xi_2 \cdot u_1(D,R) + (1-\xi_2)\cdot\Bigg(\left(1 - \lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_r\right) + \delta'\right) u_1(U,L) +$$
$$\left(\lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_r\right) - \delta'\right) \cdot \left(u_1(U,L) - \delta^B\right)\Bigg).$$

And if we take $\delta' < \frac{1}{2}\lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_r\right)$ and remember that $\lim_{k\to\infty} P^k\left(\left[E_L(-t^B)\right]^c \mid E_r\right) \geq \bar{p}$ we get an upper bound of

$$\xi_2 \cdot u_1(D,R) + (1-\xi_2)\cdot u_1(U,L) - \frac{1}{2}(1-\xi_2)\bar{p}\delta^B. \tag{7}$$

Player 1 can deviate to the strategy in which she prepares U until time $-\frac{t^B}{2}$, and then best-responds to the prepared profile. Note that

$$\lim_{k\to\infty} P^k\left(E_{1r}^c \mid \left[E_D(-t^B)\right]^c \cap \left[E_L(-t^B)\right]^c \cap E_{2r}\right) =$$
$$\lim_{k\to\infty} \frac{\xi_2}{\xi_2 + (1-\xi_2)\cdot P^k\left(\left[E_D(-t^B)\right]^c \mid \left[E_L(-t^B)\right]^c \cap E_r\right)} = 1.$$

This deviation will ensure the rational Player 1 an expected payoff bounded below by

$$\begin{aligned}
&\xi_2 \cdot \left(\left(1 - e^{-\frac{1}{2}\lambda_1 t^B}\right) u_1(D,R) + e^{-\frac{1}{2}\lambda_2 t^B} u_1(U,R)\right) + \\
&(1-\xi_2) \cdot \left(\left(1 - e^{-\frac{1}{2}\lambda_2 t^B}\right) u_1(U,L) + e^{-\frac{1}{2}\lambda_2 t^B} u_1(U,R)\right) > \\
&\xi_2 \cdot u_1(D,R) + (1-\xi_2) \cdot u_1(U,L) - \frac{1}{2}(1-\xi_2)\bar{p}\delta^B.
\end{aligned} \tag{8}$$

From (7) and (8) it follows that Player 1 has a profitable deviation, and we get a contradiction. ■

We have

$$\lim_{k\to\infty} P^k \left(\left[E_L(-t^B)\right]^c \mid E_r \right) \geq \bar{p} > 0 \qquad \text{(Claim 4.1, } -t^B < -t^A\text{)}$$
$$\lim_{k\to\infty} P^k \left(\left[E_D(-t^B)\right]^c \mid \left[E_L(-t^B)\right]^c \cap E_r \right) > 0 \qquad \text{(Claim 4.2)}$$

which together imply that

$$\lim_{k\to\infty} P^k \left(\left[E_D(-t^B)\right]^c \cap \left[E_L(-t^B)\right]^c \mid E_r \right) > 0,$$

and therefore substantial delay is induced. □

Proposition 5. *If $\xi_1, \xi_2 > 0$, both players play cutoff strategies, and Player 2's cutoff is earlier than Player 1's cutoff, then the probability of reaching an ex-post Pareto inferior outcome is*

1. *increasing in $u_1(D,R)$, and decreasing in $u_1(U,L)$ and $u_1(U,R)$,*
2. *increasing in $u_2(D,R)$ and $u_2(U,R)$, and decreasing in $u_2(U,L)$.*

Proof. In an equilibrium of the prescribed form with cutoff times $-t_1^*$ and

$-t_2^*$, where $-t_2^* \leq -t_1^*$, the following three equations must hold.

Indifference of Player 1: (9)

$$u_1(D,R) = (1 - q(t_2^* - t_1^*)) \cdot \Bigg[\left(1 - e^{-(\lambda_1+\lambda_2)t_1^*}\right) \cdot \frac{\lambda_2 u_1(U,L) + \lambda_1 u_1(D,R)}{\lambda_1 + \lambda_2} +$$
$$e^{-(\lambda_1+\lambda_2)t_1^*} \cdot u_1(U,R) \Bigg] + q(t_2^* - t_1^*) \cdot \left[\left(1 - e^{-\lambda_1 t_1^*}\right) u_1(D,R) + e^{-\lambda_1 t_1^*} u_1(U,R) \right].$$

Indifference of Player 2: (10)

$$u_2(U,L) = \left(1 - e^{-\lambda_2(t_2^*-t_1^*)}\right) u_2(U,L) + e^{-\lambda_2(t_2^*-t_1^*)} \times$$
$$\Bigg[(1-\xi_1) \cdot \Big[\left(1 - e^{-(\lambda_1+\lambda_2)t_1^*}\right) \cdot \frac{\lambda_2 u_2(U,L) + \lambda_1 u_2(D,R)}{\lambda_1 + \lambda_2} +$$
$$e^{-(\lambda_1+\lambda_2)t_1^*} u_2(U,R) \Big] + \xi_1 \cdot \left[\left(1 - e^{-\lambda_2 t_1^*}\right) u_2(U,L) + e^{-\lambda_2 t_1^*} u_2(U,R) \right] \Bigg].$$

Bayesian updating: (11)

$$q(\bar{t}) = \frac{\xi_2}{\xi_2 + (1-\xi_2)e^{-\lambda_2 \bar{t}}}.$$

Note first that (10) can be reduced to

$$u_2(U,L) = (1-\xi_1) \cdot \Big[\left(1 - e^{-(\lambda_1+\lambda_2)t_1^*}\right) \cdot \frac{\lambda_2 u_2(U,L) + \lambda_1 u_2(D,R)}{\lambda_1 + \lambda_2} + e^{-(\lambda_1+\lambda_2)t_1^*} u_2(U,R) \Big] + \xi_1 \cdot \left[\left(1 - e^{-\lambda_2 t_1^*}\right) u_2(U,L) + e^{-\lambda_2 t_1^*} u_2(U,R) \right] \tag{12}$$

In equilibrium, $-t_1^*$ is nailed down by (12), and so any change to Player 1's payoffs does not affect $-t_1^*$, and changes only $-t_2^*$. Assume that strategies defined by cutoffs $(-t_1^*, -t_2^*)$ form an equilibrium, and examine (9). Note that from the structure imposed on the payoffs it is always true that $u_1(D,R)$ is greater than the second part of the RHS of (9) (the part multiplied by $q(t_2^* - t_1^*)$). The first part of the RHS of (9) must therefore be greater than $u_1(D,R)$. If $u_1(U,L)$ becomes larger it makes the first part of the RHS even greater, and since t_1^* does not change, this must mean that $q(t_2^* - t_1^*)$ goes up, which in turns means that t_2^* becomes smaller. The probability of reaching (U,R) is simply $e^{-(\lambda_1 t_1^* + \lambda_2 t_2^*)}$, and so if t_1^* remains the same and t_2^* becomes larger, this probability becomes smaller. A similar argument shows that raising $u_1(U,R)$ also decreases this probability. When $u_1(D,R)$ increases, the LHS of (9) rises more than the RHS, and then by a similar argument the probability of reaching (U,R) increases.

Dealing with changes in Player 2's payoffs is only slightly more involved. Raising $u_2(D,R)$ makes the first part of the RHS of (12) larger, and so it must be that $e^{-(\lambda_1+\lambda_2)t_1^*}$ becomes larger as well, and this means that t_1^* becomes smaller. Looking now at (9), we see that the decrease in t_1^* has two direct meanings if t_2^* is kept constant: $q(t_2^*-t_1^*)$ increases, and $e^{-(\lambda_1+\lambda_2)t_1^*}$ and $e^{-\lambda_1 t_1^*}$ both increase. This means that each part of the RHS of (9) becomes smaller, and the weight on the second part becomes larger. Both effects lead to the RHS becoming smaller, and to offset it, $q(t_2^*-t_1^*)$ must decrease, which implies that t_2^* decreases. So both t_1^* and t_2^* decrease, so that the probability of reaching (U,R) increases. Similar arguments show that this probability decreases with $u_2(U,L)$ and increases with $u_2(U,R)$. □

Theorem 6. *1. Assume Player* 1 *is stronger than Player* 2*, and both players play cutoff strategies. Then in the limit as* $\xi_1 \to 0$ *and* $\xi_2 \to 0$*, the probability of reaching ex-post inefficiency tends to zero.*

2. Assume players are equally strong ($\Delta_{12}(u_1,u_2;\lambda_1,\lambda_2)=0$*),* $\lambda_1=\lambda_2=1$*,*[26] *and both players play cutoff strategies. Let the sequence* $(\xi_1^k,\xi_2^k)_{k=1}^{\infty}$ *be such that* $\lim_{k\to\infty}\xi_1^k=\lim_{k\to\infty}\xi_2^k=0$*, and* $\lim_{k\to\infty}\frac{\xi_2^k}{\xi_1^k}<1$*. Then in the limit as* $k\to\infty$*, the probability of reaching ex-post inefficiency tends to*

$$s_1(u_1;\lambda_1,\lambda_2)\times\lim_{k\to\infty}\frac{\xi_2^k}{\xi_1^k}\quad\left[=s_2(u_2;\lambda_1,\lambda_2)\times\lim_{n\to\infty}\frac{\xi_2^k}{\xi_1^k}\right].$$

Proof. <u>Part 1:</u> For any given ξ_1 and ξ_2, consider a SE that is defined by a pair of strategies for the rational players. Player 1 prepares U until time $-t_1^*(\xi_1,\xi_2)$, and best-responds (according to the component game's payoffs) from there on. Similarly, the rational Player 2 prepares R until time

[26] This theorem *requires* players' revision rates to be equal. This is because solving the equations involves a polynomial of degree $\frac{\lambda_1}{\lambda_2}$. Proofs for the cases where $\frac{\lambda_1}{\lambda_2}\in\{\frac{1}{4},\frac{1}{3},\frac{1}{2},2,3,4\}$ can also be found. Furthermore, for the specific case of a symmetric payoff matrix, it is possible to prove this claim for arbitrary revision rates. It is a reasonable conjecture that the theorem holds even for non-symmetric payoffs and arbitrary revision rates.

$-t_2^*(\xi_1, \xi_2)$, and best-responds (according to the component game's payoffs) from there on. In order for these two strategies to form a SE it is sufficient that three conditions are satisfied. The first is that the rational Player 2 "gives up" before Player 1 does, that is $-t_2^*(\xi_1, \xi_2) \leq -t_1^*(\xi_1, \xi_2)$.

The second condition is that Player 1 is indifferent between preparing U and D at $-t_1^*(\xi_1, \xi_2)$, conditional on the current prepared action of Player 2 being R and on her playing according to the prescribed strategy at any subsequent time. This can be written as

$$\begin{aligned} u_1(D,R) = {} & \left(1 - q\left(t_2^*(\xi_1,\xi_2) - t_1^*(\xi_1,\xi_2)\right)\right) \times \\ & \left[\left(1 - e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)}\right) \cdot \frac{\lambda_2 u_1(U,L) + \lambda_1 u_1(D,R)}{\lambda_1+\lambda_2} + \right. \\ & \left. e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)} \cdot u_1(U,R) \right] + q\left(t_2^*(\xi_1,\xi_2) - t_1^*(\xi_1,\xi_2)\right) \times \\ & \left[\left(1 - e^{-\lambda_1 t_1^*(\xi_1,\xi_2)}\right) \cdot u_1(D,R) + e^{-\lambda_1 t_1^*(\xi_1,\xi_2)} \cdot u_1(U,R) \right], \end{aligned} \tag{13}$$

where $q(\bar{t})$ is the posterior probability that Player 1 assigns to the event that Player 2 is the commitment type conditional on Player 2 not preparing L on an interval of length $\bar{t}$ and her playing a strategy that dictates preparing L on this interval, which is given by

$$q(\bar{t}) = \frac{\xi_2}{\xi_2 + (1-\xi_2)e^{-\lambda_2 \bar{t}}}. \tag{14}$$

Note that as $q(\cdot)$ is weakly increasing in $\bar{t}$, Player 1 strictly prefers preparing U before $-t_1^*(\xi_1, \xi_2)$, and strictly prefers preparing D after $-t_1^*(\xi_1, \xi_2)$.

The third condition is that the rational type of Player 2 is indifferent at $-t_2^*(\xi_1, \xi_2)$ conditional on Player 1's prepared action being U and her playing according to the prescribed strategy at any subsequent time:

$$\begin{aligned} u_2(U,L) = {} & \left(1 - e^{-\lambda_2\left(t_2^*(\xi_1,\xi_2) - t_1^*(\xi_1,\xi_2)\right)}\right) \cdot u_2(U,L) + \\ & e^{-\lambda_2\left(t_2^*(\xi_1,\xi_2) - t_1^*(\xi_1,\xi_2)\right)} \times \left[(1-\xi_1) \cdot \left[\left(1 - e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)}\right) \cdot \right. \right. \\ & \left. \frac{\lambda_2 u_2(U,L) + \lambda_1 u_2(D,R)}{\lambda_1+\lambda_2} + e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)} \cdot u_2(U,R) \right] + \\ & \left. \xi_1 \cdot \left[\left(1 - e^{-\lambda_2 t_1^*(\xi_1,\xi_2)}\right) u_2(U,L) + e^{-\lambda_2 t_1^*(\xi_1,\xi_2)} u_2(U,R) \right] \right]. \end{aligned} \tag{15}$$

This means that the rational type of Player 2 weakly prefers preparing R prior to $-t_2^*(\xi_1,\xi_2)$, and weakly prefers preparing L after $-t_2^*(\xi_1,\xi_2)$ (she strictly prefers preparing L after $-t_1^*(\xi_1,\xi_2)$). It remains to show that these three conditions can be met simultaneously (as $T\to\infty$). To see that, note first that (15) reduces to

$$\begin{aligned} u_2(U,L) = (1-\xi_1)\cdot\Big[\left(1-e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)}\right)\cdot \\ \frac{\lambda_2 u_2(U,L)+\lambda_1 u_2(D,R)}{\lambda_1+\lambda_2} + e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)}\cdot u_2(U,R)\Big] + \\ \xi_1\cdot\Big[\left(1-e^{-\lambda_2 t_1^*(\xi_1,\xi_2)}\right)u_2(U,L)+e^{-\lambda_2 t_1^*(\xi_1,\xi_2)}u_2(U,R)\Big], \end{aligned} \tag{16}$$

which implies that $t_1^*(\xi_1,\xi_2)$ only depends on ξ_1. Furthermore, from continuity:

$$\lim_{\xi_1\to 0} e^{-(\lambda_1+\lambda_2)t_1^*(\xi_1,\xi_2)} = \frac{\lambda_1\left(u_2(D,R)-u_2(U,L)\right)}{\lambda_1 u_2(D,R)+\lambda_2 u_2(U,L)-(\lambda_1+\lambda_2)u_2(U,R)}.$$

Therefore $t_1^*(\xi_1,\xi_2)$ approaches a constant as $\xi_1\to 0$, and in this equilibrium the rational Player 2 is indifferent between insisting on playing R or not along the interval $[-T,-t_1^*(\xi_1,\xi_2)]$. Knowing that $t_1^*(\xi_1,\xi_2)$ does not change with ξ_2 and converges to a constant, and looking at (13), we can immediately deduce that $q\left(t_2^*(\xi_2)-t_1^*\right)$ also converges to a constant. It follows from (14) that $t_2^*(\xi_1,\xi_2)$ tends to infinity as $\xi_1,\xi_2\to 0$. This means that for small enough ξ_2 the first condition is also satisfied, and ensures that the strategies we described form an equilibrium for small enough ξ_1 and ξ_2.

As mentioned above, when $\xi_1,\xi_2\to 0$, $t_1^*(\xi_1,\xi_2)$ approaches some constant, and $t_2^*(\xi_1,\xi_2)$ tends to infinity. This means that the limit of the expected payoffs (for the rational types) is $u(U,L)$, and the probability of reaching the Pareto inferior outcome approaches zero.

Part 2: Assume that $-t_2^*(\xi_1^k,\xi_2^k)\le -t_1^*(\xi_1^k,\xi_2^k)$.[27] We need to solve a set of four equations, namely, Equations (9), (12), (11), and $\Delta_{12}(u_1,u_2)=0$. Inputting all of these into a standard mathematical solver and then simplifying

[27] This is not without loss of generality, because the two players have different payoffs and different probabilities of being the commitment type. Nevertheless, we will show that this is the right assumption if $\lim_{k\to\infty}\frac{\xi_2^k}{\xi_1^k}<1$.

gives

$$
\begin{aligned}
e^{-t_1^*(\xi_1^k,\xi_2^k)} =& -\xi_1^k s_2(u_2)\left(\frac{u_2(U,L)-u_2(U,R)}{u_2(D,R)-u_2(U,L)}\right)+\\
&\sqrt{(1-\xi_1^k)^2 s_2(u_2)+(\xi_1^k)^2 s_2^2(u_2)\left(\frac{u_2(U,L)-u_2(U,R)}{u_2(D,R)-u_2(U,L)}\right)^2}\\
e^{-t_2^*(\xi_1^n,\xi_2^n)} =& \frac{\xi_2^k}{\xi_1^k}\times\Bigg[-\xi_1^k s_1(u_1)\left(\frac{u_1(D,R)-u_1(U,R)}{u_1(U,L)-u_1(D,R)}\right)+\\
&\sqrt{(1-\xi_1^k)^2 s_1(u_1)+(\xi_1^k)^2 s_1^2(u_1)\left(\frac{u_1(D,R)-u_1(U,R)}{u_1(U,L)-u_1(D,R)}\right)^2}\Bigg].
\end{aligned}
$$

Taking the limit as $k\to\infty$ gives us

$$
\begin{aligned}
\lim_{k\to\infty} e^{-t_1^*(\xi_1^k,\xi_2^k)} =& \sqrt{s_2(u_2)}\\
\lim_{k\to\infty} e^{-t_2^*(\xi_1^k,\xi_2^k)} =& \frac{\xi_2}{\xi_1}\times\sqrt{s_1(u_1)}.
\end{aligned}
$$

Given cutoff strategies the probability that both players never revise their strategies is $e^{-t_1^*(\xi_1^k,\xi_2^k)}\times e^{-t_2^*(\xi_1^k,\xi_2^k)}$, and this is also the probability of reaching an ex-post inefficient outcome. Because $s_1(u_1)=s_2(u_2)$ we get in the limit exactly

$$
s_1(u_1;\lambda_1,\lambda_2)\times\lim_{k\to\infty}\frac{\xi_2^k}{\xi_1^k}\quad\left[=s_2(u_2;\lambda_1,\lambda_2)\times\lim_{k\to\infty}\frac{\xi_2^k}{\xi_1^k}\right]
$$

□

Lemma 7. *In the war of attrition with Poisson arrivals (and incomplete information), for any T, all sequential equilibria have the property that there exists a cutoff time $-t^*$ such that both players exit if called to play after $-t^*$, and at least one of them exits with zero probability before $-t^*$.*

Proof. Let us denote

$$
\sigma_i(-t)=E_j\left[\sigma_i\left(\mathcal{I}_i\left(-t,\tau_i^r,\mathcal{O}_i'\right)\right)\mid -t\in\mathcal{O}_i'\right],
$$

where the expectation is taken with respect to Player j's beliefs on the possible realizations of revision opportunities for Player i up until time $-t$ (that is, it is a simple expectation derived from the definition of the Poisson process), such that $-t$ itself is a revision opportunity as well.

We wish to show that any pair of strategies played in equilibrium satisfies some kind of restricted "pairwise-monotonicity," or formally

$$(\sigma_i(-t) > 0) \wedge (-t < -t') \wedge \left(\int_{-t}^{-t'} \sigma_j(\tau) \, \mathrm{d}\tau > 0 \right) \Longrightarrow \sigma_i(-t') = 1.$$

Note that if $\sigma_i(-t) > 0$, then Player i's continuation payoff at $-t$ if not exiting is less than or equal to her payoff if exiting (the continuation payoff cannot rely on previous revision opportunities because they do not affect Player j's behavior). The continuation payoff at time $-t$ is a convex combination of (1) the expected payoff in case Player i is called to make a decision on the interval $(-t, -t']$ and exits (in which case she gets the same payoff), (2) the expected payoff in case Player j is called to make a decision on the interval $(-t, -t']$ and exits, and (3) the continuation payoff at t'. Since the probability of Player j exiting is at least $e^{-\lambda_i(t-t')} \left(1 - e^{-\lambda_j(t-t')}\right) \cdot \int_{-t}^{-t'} \sigma_j(\tau) \, \mathrm{d}\tau > 0$, the expected continuation payoff at time $-t'$ must be strictly lower than the payoff from exiting. This means that Player i exits at $-t'$, i.e., $\sigma_i(-t') = 1$.[28]

Finally, define

$$-t^* = \limsup \left\{ -t \;\middle|\; \min_i \int_{-T}^{-t} \sigma_i(\tau) \, \mathrm{d}\tau = 0 \right\}.$$

It is immediate from the definition that at least one of the players exits with zero probability before $-t^*$.

Assume (without loss of generality) that $\int_{-T}^{-t^*} \sigma_2(\tau) \, \mathrm{d}\tau \geq \int_{-T}^{-t^*} \sigma_1(\tau) \, \mathrm{d}\tau$. To see that for any $-t' > -t^*$ we have $\sigma_1(-t') = \sigma_2(-t') = 1$, consider first how the pairwise-monotonicity property works for Player 2. Let $\epsilon > 0$ be such that $\epsilon < \int_{-t^*}^{-t'} \sigma_1(\tau) \, \mathrm{d}\tau$, and let $-t \in [-T, -t^* + \epsilon)$ be such that $\sigma_2(-t) > 0$ (exists from definition of $-t^*$). This implies that $\sigma_2(-t') = 1$. This is true for arbitrary $-t' > -t^*$, so $\sigma_2(-t) = 1$ for all $-t \in (-t^*, 0]$. Now let $-t \in (-t^*, -t^* + \epsilon)$ be such $\sigma_1(-t) > 0$, and since $\int_{-t}^{-t'} \sigma_2(\tau) \, \mathrm{d}\tau > 0$ the pairwise-monotonicity property again implies $\sigma_1(-t') = 1$. □

Corollary 8. *In the war of attrition with Poisson arrivals (and incomplete information), for any T all SE payoffs can be attained by SEs in which both players use cutoff strategies.*

[28] We here use the sequential rationality property of the equilibrium; the last step would not have worked for a Nash equilibrium concept.

Proof. We wish to modify the strategies to get a new pair of strategies that are essentially the same as the old pair, and still constitute an equilibrium. Similar to the proof of Lemma 7, let

$$\sigma_i'\left(\mathcal{I}_i\left(-t, \tau_i^r, \mathcal{O}_i\right)\right) = E_j\left[\sigma_i\left(\mathcal{I}_i\left(-t, \tau_i^r, \mathcal{O}_i'\right)\right) \mid -t \in \mathcal{O}_i'\right].$$

Note that the fact that Player i could have played any continuation strategy at time $-t$ without Player j knowing about it implies that she must be indifferent between playing σ_i and σ_i'. Furthermore, σ_j is a best-response to σ_i', because nothing was changed with respect to Player j's beliefs or expected continuation payoffs. For the rest of the proof we abbreviate and write $\sigma_i'(-t)$ instead of $\sigma_i'\left(\mathcal{I}_i\left(-t, \tau_i^r, \mathcal{O}_i\right)\right)$.

Define

$$-t_i^* \equiv \inf\left\{-t \in [-T, 0] \;\middle|\; \forall -t' \in (-t, 0] : \int_{-t}^{-t'} \sigma_i'(\tau)\,\mathrm{d}\tau > 0\right\}.$$

That is, $-t_i^*$ is the earliest point from which Player i has a strictly positive probability of exiting (it could be that $-t_i^* = 0$). We know from Lemma 7 that there are only two possible equilibrium structures:

1. $-t_2^* \leq -t_1^*$, and both players exit starting from $-t_2^*$.
2. $-t_2^* > -t_1^*$, and both players exit starting from $-t_1^*$.

Without loss of generality, consider the first case, and define $-\hat{t}_2^*$ as

$$\hat{t}_2^* = t_1^* + \int_{-t_2^*}^{-t_1^*} \sigma_2'(\tau)\,\mathrm{d}\tau.$$

We claim that the strategies in which the rational type of Player 1 exits from $-t_1^*$, and the rational type of Player 2 from $-\hat{t}_2^*$, constitute a SE. To see that, note that Player 1's incentives are essentially the same, but they might have "shifted," because now the relation between the exact time and the probability is different. However, for every posterior probability of Player 2 being the rational type, the same probability that Player 2 is going to exit until time $-t_1^*$ remains the same, and so the fact that the previous pair of strategies was a SE implies that the new strategy is a best-response as well. $\square$

Corollary 9. *In a war of attrition model, assume that Player* 1 *is stronger than Player* 2*, and* $\xi_1 = 0$*. Then there exists* $\bar{\xi}_2$ *such that for every* $\xi_2 \in (0, \bar{\xi}_2)$*, the parameters induce substantial delay and inefficiency.*

Proof. The proof follows the same lines as the proof of Theorem 2, and we omit some of the technical arguments and definitions.

Let $\{T^k\}_{k=1}^{\infty}$ be a sequence of horizons such that $T^k \to \infty$ and let $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ be a corresponding sequence of equilibria. We let P^k denote the probability measure induced by Nature's moves (i.e., the lottery over types and the stochastic Poisson processes governing players' exit opportunities) and by equilibrium strategies $\hat{\sigma}^k$.

Following Corollary 8 we know that there is a sequence of equilibria in cutoff strategies, denoted by $\{\bar{\sigma}^k\}_{k=1}^{\infty}$, such that the expected payoffs from $\bar{\sigma}^k$ are the same as the expected payoffs from $\hat{\sigma}^k$. Let $\bar{\sigma}^k$ be defined by the cutoffs $\left(-\bar{t}_1^k, -\bar{t}_2^k\right)$.

If $\lim_{k\to\infty} \bar{t}_2^k = \infty$,[29] then Player 2's payoffs must approach $u_2(D, R)$. If they are not, then Player 2 can deviate to the strategy of never exiting, thus convincing Player 1 that Player 2 is the commitment type, and getting a payoff that approaches $u_2(D, R)$. This implies that Player 1's payoffs must approach $u_1(D, R)$, but then, as in the proof of Theorem 2, Player 1 can deviate to the strategy of exiting only after $-\bar{t}_2^k$ and get a payoff that is strictly greater than $u_1(D, R)$ (for small enough selection of $\bar{\xi}_2$). We reach a contradiction, implying that $\lim_{k\to\infty} \bar{t}_2^k < \infty$. Because of the same deviation, it cannot be that $\lim_{k\to\infty} \bar{t}_1^k = \infty$, and so we get that the sequence $\{\hat{\sigma}^k\}_{k=1}^{\infty}$ exhibits substantial delay.

Moving back to the sequence $\{\bar{\sigma}^k\}_{k=1}^{\infty}$, it must be that the common cutoff time, which we will denote by $-\hat{t}^k$, does not approach infinity either, and there is a positive probability of reaching it. This means that the parameters induce substantial delay, and inefficiency follows. $\square$

Corollary 10. *In a war of attrition model, assume Player* 1 *is stronger than Player* 2*; then her preferred outcome is the unique limit of the equilibrium payoff set as the probability of Player* 2 *being the commitment type approaches zero. Formally,* $\liminf_{\xi_2\to 0} \bar{\phi}^{woa}(u_1, u_2; 0, \xi_2; \lambda_1, \lambda_2) = \{u(U, L)\}$.

Proof. The existence proof is very similar to (an abbreviated version of) the proof of Theorem 6. Following Corollary 8, we need only to show that there is no other pair of cutoff strategies that forms a SE. We can rule out this possibility by showing that there are no such equilibria with $-t_1^* < -t_2^*$

[29] For convenience, and without loss of generality, we will assume throughout the proof that all the sequences we mention converge (in the broad sense).

for small enough ξ_2 and for large enough T.[30] The other case, in which $-t_2^* < -t_1^*$, is already nailed down in the calculations appearing in the proof for Theorem 6. To see that indeed there are no equilibria of the former type, suppose that Player 1 starts exiting first. Then the rational Player 2 must be indifferent at $-t_2^*$, which gives

$$u_2(U,L) = \left(1 - e^{-(\lambda_1+\lambda_2)t_2^*}\right) \cdot \frac{\lambda_2 u_2(U,L) + \lambda_1 u_2(D,R)}{\lambda_1+\lambda_2} + e^{-(\lambda_1+\lambda_2)t_1^*} \cdot u_2(U,R)$$

or

$$e^{-t_2^*} = \sqrt[\lambda_1+\lambda_2]{\frac{\lambda_1 \left(u_2(D,R) - u_2(U,L)\right)}{\lambda_1 u_2(D,R) + \lambda_2 u_2(U,L) - (\lambda_1+\lambda_2) u_2(U,R)}}. \tag{17}$$

We also know that Player 1 is indifferent at $-t_2^*$ (because she is indifferent along the interval $(-t_1^*, -t_2^*)$), that is,

$$u_1(D,R) = (1-\xi_2)\Bigg[\left(1 - e^{-(\lambda_1+\lambda_2)t_2^*}\right) \frac{\lambda_2 u_1(U,L) + \lambda_1 u_1(D,R)}{\lambda_1+\lambda_2} + \\ e^{-(\lambda_1+\lambda_2)t_2^*} u_1(U,R)\Bigg] + \xi_2 \left[\left(1 - e^{-\lambda_1 t_2^*}\right) u_1(D,R) + e^{-\lambda_1 t_2^*} u_1(U,R)\right].$$

If we define $\overline{t_2} \equiv \lim_{\xi_2 \to 0} \lim_{T \to \infty} t_2^*(\xi_2, T)$, taking the last expression to the limit gives us

$$e^{-\overline{t_2}} = \sqrt[\lambda_1+\lambda_2]{\frac{\lambda_2 \left(u_1(U,L) - u_1(D,R)\right)}{\lambda_1 u_1(D,R) + \lambda_2 u_1(U,L) - (\lambda_1+\lambda_2) u_1(U,R)}}. \tag{18}$$

Turning back to the assumption that Player 1 is stronger than Player 2, (17) together with the limit in (18) yield a contradiction, as needed when ξ_2 tends to zero. □

[30] Both t_1^* and t_2^* may be functions of ξ_2 and T. We omit this dependence in our notation except where it is crucial for understanding.