Stability vs. No Justified Envy

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Abstract

Stability and "no justified envy" are used almost synonymously in the matching theory literature. However, they are conceptually different and have logically separate properties. We generalize the definition of justified envy to environments with arbitrary school preferences, feasibility constraints, and contracts, and show that stable allocations may admit justified envy. When choice functions are substitutable, the outcome of the deferred acceptance algorithm is both stable and admits no (strong) justified envy.

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1 Introduction

Stability and "no justified envy" are used almost synonymously in the matching theory literature. However, they have different logical meanings, and they reflect different concerns of market designers. Stability is defined as the property that no coalition can profitably deviate from a matching (Gale and Shapley, 1962). This definition is motivated by the concern that following a centralized matching process, some agents may deviate together, thus hindering the implementation of the intended outcome. Stability is widely considered a key determinant of the success or failure of a centralized clearinghouse (Roth, 1990).

The concept of no justified envy was introduced by Abdulkadiroğlu and Sönmez (2003) and is akin to a fairness condition proposed by Balinski and Sönmez (1999). Justified envy arises when a single agent is convinced she is prioritized over another agent, and prefers the outcome of that other agent to her own. Consider, for example, a public-school seat-allocation scenario. Student i is prioritized over student j at school s (e.g., because she lives closer to the school, and schools prioritize students according to proximity). If j is assigned to s, while i is assigned to a school she likes less, i experiences justified envy. In this case, it may be unlikely that student i will deviate together with public school s, but it is not unlikely that she will file an appeal or even argue her case in court.[1]

Despite these different motivations, stability and no justified envy are sometimes used interchangeably, and in some situations they indeed go hand in hand.[2] In the context of school choice, where schools all have responsive preferences—such as those described by a capacity and a ranking of students—Abdulkadiroğlu and Sönmez (2003) show that a stable allocation admits no justified envy. Conversely, an individually rational and non-wasteful matching that admits no justified envy

[1]For a recent case ruling on a similar issue regarding the school-choice system in Amsterdam, see: `http://uitspraken.rechtspraak.nl/inziendocument?id=ECLI:NL:RBAMS:2015:4085&` (in Dutch, retrieved February 3, 2020).

[2]For a recent example, consider Abdulkadiroğlu et al. (forthcoming) and Dogan and Ehlers (2020) who study the same setting and describe the same property under different labels (minimal instability and justified-envy minimal, respectively).

is stable. However, the two concepts differ in more general domains, and particularly when there are feasibility constraints or contracts. These two extended domains have been a key focus of research in matching theory and market design in the past decade.

An example of how feasibility constraints draw a wedge between stability and no justified envy comes from no-spouse and anti-nepotism employment policies. These policies prevent family members from being hired by or admitted to the same institution. While illegal in most states of the United States today (as they may be considered a form of marital status discrimination), these kind of rules were prevalent in the past in the private and the public sector,[3] and they are still common in other countries.[4] When such employment policies are in place, an institution may rank two family members as the two most desirable candidates, but end up hiring only one of them, together with another, lower-ranked candidate. In this case, the family member not hired (arguably) experiences justified envy toward the less qualified candidate who was hired.[5]

Studying justified envy in a more general environment (with arbitrary preferences, feasibility constraints, and contracts) is not straightforward. It is not always clear who is prioritized over whom, and as a consequence what type of envy is "justified." Consider, for example, a college that uses one ranking for admissions and another ranking to determine how to allocate funding. It is not necessarily clear that the first student in line for funding is prioritized (in any meaningful sense) over any of the other students. She may be the last student in the admissions ranking. Moreover, both rankings may not be observable to an outside analyst.

Our approach is to first infer a priority relation from schools' observable choices. Schools' choices (represented by their choice functions) are determined by their

[3]For a survey of case studies and a historical overview, see Chandler et al. (2002).

[4]In Israel, for example, anti-nepotism employment regulations are relatively common and exist in all branches of public service (Koch Davidovich, 2010).

[5]In the context of multi-dimensional constraints, Delacrétaz et al. (2019) note that a weak form of no justified envy is independent of stability.

preferences and potentially also by feasibility constraints imposed by a regulator.[6] Choice functions are therefore a general way to describe both incentives and policy goals. We derive two priority relations, and both coincide with schools' rankings in the case of responsive preferences (but in general they are not total orders over students). Second, we generalize the definition of justified envy to cases where "justification" is given by a priority relation, like the ones we infer. We then study the connection between stability and no justified envy in a model of matching with contracts (as in Hatfield and Milgrom, 2005) and constraints (see, e.g., Kamada and Kojima, 2019, and the references therein).

In this general model, the implication from stability to no justified envy depends on the properties of the schools' choice functions and on the strength of the derived priority relation. For example, if the priority relations are empty for all schools, there can be no justified envy.

When schools only have one type of position, and schools' choices are constrained by regulation or any other source of feasibility constraints, a stable allocation may admit weak justified envy, i.e., based on the more permissive priority relation we infer. In this case, no stable allocation admits strong justified envy, i.e., based on the more restrictive priority relation we infer.

However, in the presence of multiple contract terms, a stable allocation may even admit strong justified envy. In some cases, this is true for *all* stable outcomes. For substitutable choice functions, this cannot happen. Substitutability is a well-known property of choice functions, roughly stating that removing a contract from the choice menu does not make other contracts less desirable. It has been used by Hatfield and Milgrom (2005) to prove the existence of a stable allocation in many-to-one matching with contracts. When choice functions are substitutable, a stable allocation that admits no strong justified envy always exists. One such allocation can be reached using the student-proposing deferred acceptance (DA) algorithm (Gale and Shapley, 1962), providing yet another justification for using this mechanism.

[6] Conceptually related is the literature on revealed preference in matching markets (See Chambers and Echenique, 2016, Chapter 10 and references therein).

The paper is structured as follows. Section 2 presents the model. Section 3 introduces our derivation of the two priority relations and presents results for generalized school-choice settings (without contracts). Section 4 covers the matching-with-contracts environment. Section 5 concludes.

2 Model

We begin by introducing a few definitions that will help us describe many-to-one matching markets with arbitrary preferences, feasibility constraints, and contracts.

Let I be a non-empty finite set of students, S a non-empty finite set of schools, and T a non-empty finite set representing potential contract terms between students and schools. We denote the set of contracts by $X = I \times S \times T$. For a contract $x = (i, s, t) \in X$, we denote by x_I the student that participates in this contract, i, and by x_S the school that participates in this contract, s. For any subset of contracts $Y \subseteq X$, we denote the subset of contracts in which a given student $i \in I$ appears by $Y_i := \{x \in Y \mid x_I = i\}$, and the subset of contracts in which a given school $s \in S$ appears by $Y_s := \{x \in Y \mid x_S = s\}$. Whenever Y_i is a singleton and there is no risk of confusion, we also refer to the single contract as Y_i.

Each student i has a strict preference order $\succ_i$ over X_i and being unmatched (denoted by $\emptyset$). When explicitly writing a student's preference, we sometimes omit contracts less preferred to $\emptyset$. Each school s has a strict preference order $\succ_s$ over subsets of X_s (including the empty set). We assume throughout that schools never rank a subset that contains two or more contracts with the same student above the empty set.

Each school is also subject to an exogenously given constraint that takes the form of a collection of sets of contracts $\mathcal{F}_s \subseteq 2^{X_s}$. We say that a subset $Y \subseteq X_s$ is **feasible** for s if $Y \in \mathcal{F}_s$, and that it is infeasible otherwise. We assume that the empty set is feasible, $\emptyset \in \mathcal{F}_s$. We denote by Ch_s the choice function induced by

$\succ_s$ and[7] $\mathcal{F}_s$:

$$\forall X' \subseteq X, Ch_s(X') := \max_{\succ_s} \{Y \subseteq X'_s \mid Y \in \mathcal{F}_s\}.$$

We refer to the tuple $\left(I, S, T, \{\succ_i\}_{i\in I}, \{\succ_s\}_{s\in S}, \{\mathcal{F}_s\}_{s\in S}\right)$ as a **market**. We now reproduce a few well-known properties of choice functions.

Definition 2.1. The choice function Ch_s is **responsive** if there exists $q_s \in \mathbb{N}$ and a ranking of the "acceptable" contracts in X_s: $x_1 >_s x_2 >_s \cdots >_s x_n$, such that for any $X' \subseteq X$ the chosen contracts are the q_s maximal (acceptable) contracts in X' according to $>_s$, and multiple contracts with the same student are not chosen twice. If fewer than q_s such contracts are available, all of them are chosen. We say that school s has **responsive preferences** if, in the absence of feasibility constraints, its preferences induce a responsive choice function.

Definition 2.2 (Hatfield and Milgrom, 2005)**.** Let R_s be the rejection function defined by $R_s(X') = X' \setminus Ch_s(X')$. The choice function Ch_s is **substitutable** if for all subsets $X' \subseteq X'' \subseteq X$ we have $R_s(X') \subseteq R_s(X'')$.

An **allocation** is a collection of contracts, $Y \subseteq X$, where no student appears in multiple contracts, i.e., $\forall i \in I, |Y_i| \leq 1$. An allocation is feasible if $Y_s \in \mathcal{F}_s$ for each $s \in S$.

We say that student i blocks allocation Y if $\emptyset \succ_i Y_i$. Similarly, school s blocks allocation Y if $Ch_s(Y_s) \succ_s Y_s$. We say that a coalition consisting of students I' and school s blocks allocation Y if there exists a non-empty subset of contracts $X' \subseteq I' \times \{s\} \times T$ such that $\forall i \in I', X'_i \succ_i Y_i$, and $X' \subseteq Ch_s(Y_s \cup X')$. Allocation Y is **stable** if it is not blocked by any student, school, or coalition.

In general, defining stability in the presence of constraints is not straightforward. A key difficulty is that some blocking pairs (or coalitions) may lead to infeasible allocations. We address this issue by integrating the constraints into the choice function, practically ruling out this kind of blocking coalitions. Similar

[7] As schools' choice functions are derived from strict preferences and feasibility constraints, they automatically satisfy the irrelevance of rejected contracts condition (Aygün and Sönmez, 2013).

solutions are suggested by Kamada and Kojima (2017) for markets with distributional constraints, and by Ehlers et al. (2014) for school choice with upper and lower bounds.

3 Generalized School Choice

In this section we only consider the case of $|T| = 1$ (the "generalized school-choice settings"). We omit any reference to the (single) term involved in the contracts. Furthermore, for any school s we abuse notation and treat $\succ_s$ as if it was an order over subsets of students, and treat Ch_s as if its domain and range were subsets of students. We return to the general setting in Section 4.

3.1 Priority Relations for Arbitrary School Preferences

We define two extensions of the priority relation used when discussing school choice with responsive preferences, which we refer to as weak priority and strong priority. The names reflect that the strong priority relation is more demanding than the weak priority relation. We derive the two priority relations from the choice function and feasibility constraints. In our definition we intentionally avoid directly using the preferences of schools, as these may not be observable.

Definition 3.1. For any two students $i, j \in I$ and school $s \in S$, we say that i is **strongly prioritized** over j at s, and write $i \gg_s j$, if

- $\forall I' \subseteq I$, $j \in Ch_s(I' \cup \{i, j)\}) \Rightarrow i \in Ch_s(I' \cup \{i, j\})$, and
- $\exists I' \subseteq I$ such that $i \in Ch_s(I' \cup \{i, j\})$ and $j \notin Ch_s(I' \cup \{i, j\})$.

In words, student i is strongly prioritized over student j if whenever the school chooses j while i is available, the school chooses i as well, and the opposite does not hold (i.e., there exists a case where both students are available but only i is chosen). The strong priority relation (at a certain school) is an asymmetric binary relation over students. This relation is transitive if Ch_s is substitutable, but otherwise it may be non-transitive (see Appendix A).[8]

Definition 3.2. For any two students $i, j \in I$ and school $s \in S$, we say that i is **weakly prioritized** over j at s, and write $i \gg_s^w j$, if

- $\forall I' \subseteq I$, $j \in Ch_s(I' \cup \{i,j\}) \Rightarrow i \in Ch_s(I' \cup \{i,j\})$ or $(Ch_s(I' \cup \{i,j\}) \setminus \{j\}) \cup \{i\} \notin \mathcal{F}_s$, and
- $\exists I' \subseteq I$ such that $i \in Ch_s(I' \cup \{i,j\})$ and $j \notin Ch_s(I' \cup \{i,j\})$.

Simply put, the new addition to the first condition allows j to be chosen even when i is not, as long as substituting i for j results in an infeasible subset.[9] The weak priority relation is motivated by the fact that a school's true preferences, as observed through choices, are masked by the feasibility constraints.[10] If those feasibility constraints are publicly known, one can infer the priorities expressed in the unobservable preference relation that induced the observable choice function.[11]

[8]One may be tempted to consider "conditional priority" along the following lines. Imagine a school with one slot reserved for minorities and a ranked list of students, some of whom belong to the minority population. According to our definition of strong priority, while the top minority student is prioritized over all other students who are ranked below the number of available places, the second from the top minority student is already not prioritized over any of the students ranked above her, nor are any of them prioritized over her. However, conditional on the top minority student not being available, the second from the top minority student seems to have a justified claim over the slot reserved for minorities. We note that such motivation is more removed from the notions of "priority" or "justification" and closer to the notion of stability, since it takes into account coalitional deviations from a proposed allocation.

[9]This definition resembles Kamada and Kojima's (2019) discussion on weaker and stronger notions of fairness under feasibility constraints. In their paper feasibility constraints are used to weaken the fairness concept that is based on a *given* priority order, whereas we use feasibility constraints to weaken the requirements for priority.

[10]A similar consideration (of whether envy is justified when a student envies something that is impossible for her to get) motivates Ehlers and Morrill (2020) to weaken the stability notion.

[11]It is indeed possible to define an even weaker priority relation that says that i is prioritized over j at s if there exists *some* preference relation $\tilde{\succ}_s$ that together with the feasibility constraints induces the same observed choices, and that absent feasibility constraints chooses i whenever j is chosen.

Clearly, $i \gg_s j \Rightarrow i \gg_s^w j$, which means that more pairs of students are comparable according to the weak priority relation. Example 3.3 compares the priority relations we defined with the rankings of students under responsive preferences.

Example 3.3 (Responsive preferences)**.** Suppose that school s has responsive preferences with a capacity q_s and ranking $i_1 > \cdots > i_n$, and $\mathcal{F}_s = 2^I$. Then for any $q_s < t_2 \leq n$ and any $t_1 < t_2$, we have that $i_{t_1} \gg_s i_{t_2}$. Also, any acceptable student i_t is prioritized over any unacceptable student $j \in I \setminus \{i_1, \ldots, i_n\}$. This corresponds to our natural interpretation of the school's ranking as a priority relation, with the caveat that the school's choice data does not reveal any information about comparisons between the top q_s students (who are all guaranteed a place in the school).[12]

Example 3.4 shows that the weak priority relation can sometimes be strictly weaker than the strong priority relation.

Example 3.4. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s\}$. School s has responsive preferences with a capacity of 2, and it ranks i_1 above i_2 above i_3 above i_4. The feasibility constraint $\mathcal{F}_s = 2^I \setminus \{\{i_1, i_2\}, \{i_1, i_2, i_3\}, \{i_1, i_2, i_4\}, I\}$ reflects the fact that i_1 and i_2 cannot be admitted simultaneously. With these preferences and constraints, $i_2 \not\gg_s i_4$, since $i_4 \in Ch_s(\{i_1, i_2, i_4\})$ but $i_2 \notin Ch_s(\{i_1, i_2, i_4\})$. However, $i_2 \gg_s^w i_4$. To see that, note that the only subset from which i_4 is chosen and i_2 is rejected is $\{i_1, i_2, i_4\}$. In this case, the choice is $\{i_1, i_4\}$, and substituting i_2 for i_4 results in an infeasible set of students.

Finally, we say that student i has **strong (weak) justified envy** toward student j at allocation Y if i prefers the school j is assigned to, $[Y_j]_S$, to her own assignment, and she is strongly (weakly) prioritized over j at that school. We say

[12]Incidentally, using our strong priority relation simplifies the definition of Ergin's (2002) acyclity condition. A market contains a cycle if there exist $i, j, k \in I$ and $a, b \in S$ such that $i \gg_a j \gg_a k \gg_b i$. Ergin's main result is that when schools' choice functions are responsive, a fair and efficient allocation is guaranteed to exist regardless of student preferences if and only if the market contains no cycle.

that Y **admits no strong (weak) justified envy** if there are no students i and j such that i has strong (weak) justified envy toward j at Y.

Unlike other definitions of justified envy that appear in the literature (e.g., Sönmez, 2013), our definition does not make use of an exogenously given priority relation, and instead uses the *revealed* priority. This makes our approach more flexible and appropriate for a variety of environments and preferences. The example of Educational Option programs in the New York City High School Match (Abdulkadiroğlu et al., 2005) provides a stark illustration of the difference between the two approaches. Educational Option programs are allowed to individually choose students for half of their seats, subject to the restriction that 16 percent be allocated to top performers in a standardized English Language Arts (ELA) exam, 68 percent to middle performers, and 16 percent to lower performers. Suppose a program s ranks students solely based on their ELA score, and chooses subsets of students subject to the above constraint. If student i has a better ELA score than student j, but student j has the highest score among "low performers," a possible outcome is that j gets admitted to s while i is not. In this case, a definition based on students' scores will deem i's envy towards j justified. By contrast, by our definition j is prioritized over i at s, since j is always chosen by s when she is available, and therefore i's envy is not justified.

It is worthwhile to mention that extending the concept of no justified envy to arbitrary preferences is already enough to set it apart from stability, as is demonstrated by Example 3.5.

Example 3.5. Let $I = \{i_1, i_2, i_3, i_4\}$, and $S = \{s\}$. School s has two seats, and its preferences are given by

$$\{i_1, i_3\} \succ_s \{i_2, i_4\} \succ_s \{i_3, i_4\} \succ_s \{i_1, i_4\} \succ_s \{i_2, i_3\} \succ_s$$
$$\{i_1, i_2\} \succ_s \{i_1\} \succ_s \{i_2\} \succ_s \{i_3\} \succ_s \{i_4\} \succ_s \emptyset.$$

There are no feasibility constraints ($\mathcal{F}_s = 2^I$). Here $Ch_s(\{i_1, i_3, i_4\}) = \{i_1, i_3\}$, so i_4 is not (strongly or weakly) prioritized over i_3. Furthermore, whenever i_2 and i_4

are both available, either both or none of them are chosen. This means that i_4 is also not (strongly or weakly) prioritized over i_2.

Suppose students' preferences are such that i_1 finds s unacceptable, but i_2, i_3, and i_4 find s acceptable. Then the allocation $\{(i_2, s), (i_3, s)\}$ is individually rational,[13] non-wasteful,[14] and admits no justified envy, but it is not stable. This stands in contrast to the characterization result that holds for responsive preferences.

3.2 Justified Envy under Feasibility Constraints

We use feasibility constraints to model any technical rules or regulations that prohibit schools from choosing certain subsets of students independently of schools' preferences. Apart from the anti-nepotism employment policies mentioned in Section 1, feasibility constraints may also arise from affirmative action policies that impose bounds on the number of chosen students from each population group (Abdulkadiroğlu, 2005). In the presence of feasibility constraints, a stable allocation may admit weak justified envy. For example, if we take I and S as in Example 3.4 and let s be acceptable to all students except student i_3, then $\{(i_1, s), (i_4, s)\}$ is the unique stable allocation, and i_2 has weak justified envy toward i_4 at this allocation. In fact, as Proposition 3.6 below demonstrates, whenever the two relations do not coincide, there exists a profile of student preferences that induces a *unique* stable allocation that admits weak justified envy.

Proposition 3.6. *In generalized school-choice settings, if for some school s the weak priority relation does not coincide with the strong priority relation, then there exists a profile of student preferences that induces a unique stable allocation that admits weak justified envy.*

Proof. If for school s, $\gg_s^w$ is different from $\gg_s$, then there exist students i and j such that $i \gg_s^w j$, but $i \not\gg_s j$. We will prove the slightly stronger claim that in

[13] We say that allocation Y is individually rational if for every student i, $Y_i \succ_i \emptyset$, and for every school s, $Ch_s(Y_s) = Y_s$.

[14] We say that student i claims an empty seat at school s under allocation Y if $s \succ_i Y_i$ and $Ch_s(Y_s \cup \{i\}) = Y_s \cup \{i\}$. Allocation Y is non-wasteful if no student claims an empty seat at any school under Y.

this case there exists a profile of student preferences that induces a unique stable allocation at which i has weak justified envy toward j.

Since $i \gg_s^w j$ and $i \not\gg_s j$, there exists $I' \subseteq I$ such that $j \in Ch_s(I' \cup \{i,j\})$ but $i \notin Ch_s(I' \cup \{i,j\})$. For each $i' \in I' \cup \{i,j\}$, let $\succ'_{i'}$ be such that s is the only acceptable school, and for each $i' \in I \setminus (I' \cup \{i,j\})$, let $\succ'_{i'}$ be such that no school is acceptable. The unique stable allocation is

$$Y = \{(i', s) \mid i' \in Ch_s(I' \cup \{i,j\})\},$$

at which j is matched with s, and i is not. Thus, i has weak justified envy toward j. □

We conclude this section by extending Abdulkadiroğlu and Sönmez's (2003) observation on responsive preferences, and present a close relation between stability and no strong justified envy in generalized school-choice settings. We note that the following result does not impose any structure on the schools' preferences.

Proposition 3.7. *In generalized school-choice settings, a stable allocation admits no strong justified envy.*

Proof. Let Y be a stable allocation. Suppose that $Y_j = s \succ_i Y_i$. By stability, $i \notin Ch_s(Y_s \cup \{i\})$. By stability and the irrelevance of rejected contracts property, $j \in Ch_s(Y_s) = Ch_s(Y_s \cup \{i\})$. Thus i does not have strong priority over j at s. □

4 Multiple Contract Terms

We now return to the general matching-with-contracts model. Before we can define priority and justified envy for this environment, we need to consider the issue of interpersonal comparisons of contracts. When defining contracts, Hatfield and Milgrom (2005) do not consider interpersonal comparisons (such as envy) and consequentially their framework allows for a very general interpretation of terms of employment. For example, Juliet's terms of employment at a specific firm could be that she gets a salary of $150k and is allowed to bring a guest, but only if his name is Romeo, to the annual office party. This works out just great for Juliet. However, this kind of contract is not necessarily envied by Desdemona, who cannot enjoy the office party with Romeo, fearing that they would both suffer the wrath of her jealous husband Othello. Since we are interested in interpersonal comparisons, we look at settings where terms are comparable (e.g., getting a certain scholarship, being admitted to a certain study track, bringing a companion to the annual office party, and so on).

With this caveat in mind, we extend the definition of strong priority by referring to specific terms of employment that are relevant for both i and j.

Definition 4.1. For any two students $i, j \in I$ and school $s \in S$, we say that i is **strongly prioritized** over j at s, and write $i \gg_s j$, if

- $\forall t \in T$, $\forall X' \subseteq X$, $(j, s, t) \in Ch_s(X' \cup \{(i, s, t), (j, s, t)\}) \Rightarrow Ch_s(X' \cup \{(i, s, t), (j, s, t)\}) \cap X_i \neq \emptyset$, and
- $\exists t \in T, \exists X' \subseteq X$ such that $(i, s, t) \in Ch_s(X' \cup \{(i, s, t), (j, s, t)\})$ and $Ch_s(X' \cup \{(i, s, t), (j, s, t)\}) \cap X_j = \emptyset$.

In words, if j is chosen with certain terms and there is a contract available with i under the same terms, then i is also chosen (under these terms, or others).[15] This definition is motivated by the idea that a student who is prioritized by the school will always be chosen to fill a specific position before a student over whom she is prioritized, unless the former is chosen to fill a different position.[16] For example, when contracts represent different study programs at a certain college, then a situation in which j is admitted to the economics program, and i is an applicant

[15] This definition can be interpreted as the dual of the definition of fair branch priorities induced by a choice function in Sönmez (2013). Sönmez asks whether a choice function violates a given priority order, whereas we first derive a priority from a given choice function.

[16] While Definition 4.1 is agnostic with regard to the interpretation of terms, an alternative definition of strong priority that assumes that terms are ordered will result in a (weakly) finer priority relation. Such a relation could be more adequate for studying environments in which terms are naturally ordered, such as college admissions with funding (Hassidim et al., forthcoming) or cadet-branch matching (Sönmez, 2013; Sönmez and Switzer, 2013; Jagadeesan, 2019).

to the same program, indicates that i should be admitted to *some* program at that college (but, if i applied to multiple programs, not necessarily to the economics program).

As before, we say that student i has **strong justified envy** toward student j at allocation Y if i prefers the school and terms to which j is assigned to her own assignment, and she is strongly prioritized over j at that school. Formally, suppose $(j, s, t) \in Y$, then i has strong justified envy toward j at Y if $(i, s, t) \succ_i Y_i$ and $i \gg_s j$.

Example 4.2 (Stable allocation with strong justified envy under substitutable choice functions)**.** Let $I = \{i, j, k\}$ and $S = \{s\}$. $T = \{\mathrm{f}, \mathrm{nf}\}$ represents funded and non-funded positions in the school. Denote $x_{i'}^{t} \equiv (i', s, t)$. The school has responsive preferences with capacity 2, and it prefers not to give funding, but cares lexicographically more about students' identities. In particular, it uses the ranking

$$x_i^{\mathrm{nf}} > x_i^{\mathrm{f}} > x_j^{\mathrm{nf}} > x_j^{\mathrm{f}} > x_k^{\mathrm{nf}} > x_k^{\mathrm{f}}.$$

There are no feasibility constraints: $\mathcal{F}_s = X_s$. The induced choice function is substitutable, and $i \gg_s k$.

Suppose that student i's preferences are given by $x_i^{\mathrm{f}} \succ_i x_i^{\mathrm{nf}} \succ_i \emptyset$, student j finds all contracts unacceptable, and student k's preferences are $x_k^{\mathrm{f}} \succ_k \emptyset$. There are two stable allocations: $Y_1 = \left\{x_i^{\mathrm{f}}, x_k^{\mathrm{f}}\right\}$, which is the student-optimal stable allocation and which admits no strong justified envy, and $Y_2 = \left\{x_i^{\mathrm{nf}}, x_k^{\mathrm{f}}\right\}$, which is the school-optimal stable allocation and at which i has strong justified envy toward k.

While Example 4.2 may seem a bit contrived, it does resemble many existing college admissions environments, in which funding levels may vary between students, and not necessarily be based on their priority for the purpose of admission. The example provides simpler college preferences than are often used in practice, since in most college admissions scenarios funding is limited, which results in a non-substitutable choice function (see, e.g., Hassidim et al., 2017).

Theorem 4.3 below shows that for substitutable choice functions, while some stable allocations may induce strong justified envy, there always exists at least one stable allocation that does not admit strong justified envy. Furthermore, such a stable allocation can be found using the student-proposing DA.[17]

Theorem 4.3. *If all schools' choice functions are substitutable, then there exists a stable allocation that admits no strong justified envy, and one such allocation can be reached using the student-proposing DA.*

Proof. We consider the cumulative offer process, which, in the domain of substitutable choice functions, produces the student-optimal stable allocation (Hatfield and Milgrom, 2005). In the cumulative offer process, students propose as in DA, but schools get to choose from all the offers they received in the current or any previous round.

Let Y be the stable allocation that results from the cumulative offer process. Suppose that some student i has strong justified envy toward another student j in Y, and that $Y_j = (j, s, t)$.

Since i envies j, and Y is stable, it must be that $(i, s, t) \notin Ch_s (Y \cup \{(i, s, t)\})$. But, since i has strong priority over j at s, it must be that $Ch_s (Y \cup \{(i, s, t)\}) \cap X_i \neq \emptyset$. Therefore there exists $t' \neq t$ such that $(i, s, t') \in Ch_s (Y \cup \{(i, s, t)\})$, and this implies that $(i, s, t') \in Y$, meaning i is assigned to s under different terms than j.

Since i is eventually assigned the contract (i, s, t') and she envies j, it must be that during the cumulative offer process i proposed to s the contract (i, s, t) and it was rejected. Let Y'_s denote the set of contracts that were available to s during the run of the cumulative offer process when (i, s, t) was first rejected. Note that Y'_s may contain contracts with i that were already rejected (and, by substitutability, are also rejected from Y'_s), and it may also contain contracts with j (at most one of which is not rejected). However, it cannot contain (j, s, t). The reason is that (j, s, t) is not rejected from a set that includes Y'_s (the set of offers available when

[17] Even when the school-proposing DA results in a stable allocation, this allocation may admit strong justified envy. See Example 4.2.

the process terminates). Thus, by substitutability, had Y'_S contained (j, s, t), this contract would have been chosen from Y'_S, in contradiction to $i \gg_s j$.

Now consider the set of all contracts available to s at the end of the cumulative offer process, and denote it by Y''_s. We know that $(j, s, t) \in Y_s = Ch_s(Y''_s)$. Since $Y'_s \subseteq Y''_s$, substitutability implies that $(j, s, t) \in Ch_s(Y'_s \cup \{(j, s, t)\})$. Now $i \gg_s j$ implies that $Ch_s(Y'_s \cup \{(j, s, t)\}) \cap X_i \neq \emptyset$, which means that i was assigned one of the contracts she proposed. And, by substitutability, we get that the same contract should have been chosen from Y'_s as well, a contradiction. □

While Theorem 4.3 equally applies in the presence of feasibility constraints and in their absence, it should be noted that the theorem's applicability may be limited by such constraints in many cases. That is, given preferences that would induce a substitutable choice function absent any feasibility constraints, imposing the constraints on it may result in a non-substitutable choice function. This is, for example, the case with budget constraints (Mongell and Roth, 1986) and with capacity constraints (Romm, 2014, Lemma 1).[18]

Finally, we observe that outside of the substitutable preferences domain, Theorem 4.3 fails to hold. Appendix A provides an example of a market with a single school whose choice function satisfies the law of aggregate demand (Hatfield and Milgrom, 2005), bilateral substitutability (Hatfield and Kojima, 2010), and substitutable completability (Hatfield and Kominers, 2015), where the *unique* stable allocation induces strong justified envy. This observation has important implications, as the two latter generalizations of substitutability are used extensively in studies of matching market design and in practice. The matter of whether or not a result similar to Theorem 4.3 holds for choice functions that satisfy unilateral substitutability (Hatfield and Kojima, 2010) turns out to be more elusive, and it is left open.

[18]See also Kojima et al. (2020) for a discussion on the effects of constraints on substitutability in a matching-with-salaries environment.

5 Discussion

The elimination of justified envy is an important concept that embodies basic fairness considerations and that deserves the attention of market designers. Abdulkadiroğlu and Sönmez (2003) observed that it is tightly related to the absence of blocking pairs in the Boston Public Schools' new matching system, and showed that stability implies no justified envy in the context of school choice. This contribution inspired a large literature that employs this connection to study stable mechanisms and their fairness properties.

As demonstrated in this paper, when constraints or contracts are present, the two concepts of stability and no justified envy become independent. While we do not take a stand regarding the desirability of either of these properties, we note that this independence opens the door for designing market mechanisms that satisfy only one of these two properties. For example, Biró et al. (2020) study college admissions in Hungary, where the current mechanism (a variant of DA) is both stable and admits no justified envy. They show that policymakers could increase the number of admitted students by selecting stable allocations that admit justified envy.

When choice functions are substitutable, using the deferred acceptance algorithm does lead to a stable allocation that also admits no justified envy. This may explain why, in many practical situations, no wedge between stability and no justified envy has been observed. It remains an open problem to identify a bigger family of choice functions that ensures the existence of a stable allocation that admits no justified envy.[19] Similarly, the problem of finding an extensive family of preferences and feasibility constraints that induce substitutable choice functions also remains open.

[19] Biró et al. (2020, Proposition 5) present a similar result for certain college admissions scenarios, where schools' choice functions are non-substitutable.

References

Atila Abdulkadiroğlu. College admissions with affirmative action. *International Journal of Game Theory*, 33(4):535–549, 2005.

Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

Atila Abdulkadiroğlu, Parag A. Pathak, and Alvin E. Roth. The new york city high school match. *American Economic Review*, 95(2):364–367, 2005.

Atila Abdulkadiroğlu, Yeon-Koo Che, Parag A. Pathak, Alvin E. Roth, and Olivier Tercieux. Efficiency, justified envy, and incentives in priority-based matching. *American Economic Review: Insights*, forthcoming.

Orhan Aygün and Tayfun Sönmez. Matching with contracts: Comment. *American Economic Review*, 103(5):2050–2051, 2013.

Michel Balinski and Tayfun Sönmez. A tale of two mechanisms: Student placement. *Journal of Economic Theory*, 84(1):73–94, 1999.

Péter Biró, Avinatan Hassidim, Assaf Romm, Ran I. Shorrer, and Sándor Sóvágó. Need vs. merit: The large core of college admissions markets. Working paper, 2020.

Christopher P. Chambers and Federico Echenique. *Revealed preference theory*, volume 56. Cambridge University Press, 2016.

Timothy D. Chandler, Rafael Gely, Jack Howard, and Robin Cheramie. Spouses need not apply: The legality of antinepotism and no-spouse rules. *San Diego Law Review*, 39:31–78, 2002.

David Delacrétaz, Scott D. Kominers, and Alexander Teytelboym. Matching mechanisms for refugee resettlement. Working paper, 2019.

Battal Dogan and Lars Ehlers. Robust minimal instability of the top trading cycles mechanism. Available at SSRN: https://ssrn.com/abstract=3555762, 2020.

Lars Ehlers and Thayer Morrill. (il) legal assignments in school choice. *Review of Economic Studies*, 87(4):1837–1875, 2020.

Lars Ehlers, Isa E. Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim. School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic Theory*, 153:648–683, 2014.

Haluk I. Ergin. Efficient resource allocation on the basis of priorities. *Econometrica*, 70(6):2489–2497, 2002.

David Gale and Lloyd S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.

Avinatan Hassidim, Assaf Romm, and Ran I Shorrer. Redesigning the israeli psychology master's match. *American Economic Review*, 107(5):205–209, 2017.

Avinatan Hassidim, Assaf Romm, and Ran I. Shorrer. The limits of incentives in economic matching procedures. *Management Science*, forthcoming.

John W. Hatfield and Fuhito Kojima. Substitutes and stability for matching with contracts. *Journal of Economic Theory*, 145(5):1704–1723, 2010.

John W. Hatfield and Scott D. Kominers. Hidden substitutes. In *Proceedings of the 16th ACM Conference on Economics and Computation*, page 37, 2015.

John W. Hatfield and Paul R. Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, 2005.

Ravi Jagadeesan. Cadet-branch matching in a Kelso–Crawford economy. *American Economic Journal: Microeconomics*, 11(3):191–224, 2019.

Sangram Vilasrao Kadam. Unilateral substitutability implies substitutable completability in many-to-one matching with contracts. *Games and Economic Behavior*, 102:56–68, 2017.

Yuichiro Kamada and Fuhito Kojima. Stability concepts in matching under distributional constraints. *Journal of Economic theory*, 168:107–142, 2017.

Yuichiro Kamada and Fuhito Kojima. Fair matching under constraints: Theory and applications. Working paper, 2019.

Flora Koch Davidovich. Employing family relatives in the civil service: A comparative review. Technical report, Knesset Research and Information Center, 2010. (in Hebrew).

Fuhito Kojima, Ning Sun, and Ning Neil Yu. Job matching under constraints. *American Economic Review*, 110(9):2935–2947, 2020.

Susan J. Mongell and Alvin E. Roth. A note on job matching with budget constraints. *Economics Letters*, 21:135–138, 1986.

Assaf Romm. Implications of capacity reduction and entry in many-to-one stable matching. *Social Choice and Welfare*, 43(4):851–875, 2014.

Alvin E. Roth. New physicians: A natural experiment in market organization. *Science*, 250(4987):1524–1528, 1990.

Tayfun Sönmez. Bidding for army career specialties: Improving the ROTC branching mechanism. *Journal of Political Economy*, 121(1):186–219, 2013.

Tayfun Sönmez and Tobias B. Switzer. Matching with (branch-of-choice) contracts at the United States Military Academy. *Econometrica*, 81(2):451–488, 2013.

A Additional Proofs and Examples

A.1 Transitivity of the Strong Priority Relation

We first look at the case of substitutable choice functions, and then provide an example for non-transitivity with a non-substitutable choice function.

Lemma A.1. *If Ch_s is substitutable, then $\gg_s$ is transitive.*

Proof. Suppose that $i \gg_s j$ and $j \gg_s k$; we need to show that $i \gg_s k$. For the first requirement, let $I' \subseteq I$ be such that $k \in Ch_s(I' \cup \{i, k\})$. If $j \in I'$ then from $j \gg_s k$ we get that $j \in Ch_s(I' \cup \{i,k\})$, and from $i \gg_s j$ we get that $i \in Ch_s(I' \cup \{i,k\})$. Otherwise, let $I'' = I' \cup \{j\}$. If $k \notin Ch_s(I'' \cup \{i,k\})$; then from the irrelevance of rejected contracts property it follows that $j \in Ch_s(I'' \cup \{i,k\})$, from $i \gg_s j$ that $i \in Ch_s(I'' \cup \{i,k\})$, and from substitutability that $i \in Ch_s(I' \cup \{i,k\})$. If $k \in Ch_s(I'' \cup \{i,k\})$, then from $j \gg_s k$, we get $j \in Ch_s(I'' \cup \{i,k\})$, from $i \gg_s j$, we get $i \in Ch_s(I'' \cup \{i,k\})$, and from substitutability, we get $i \in Ch_s(I' \cup \{i,k\})$.

For the second requirement, let $I' \subseteq I$ be such that $i \in Ch_s(I' \cup \{i,j\})$ and $j \notin Ch_s(I' \cup \{i,j\})$. Consider $I'' = I' \cup \{k\}$. From substitutability, $j \notin Ch_s(I'' \cup \{i,j\})$, and so from $j \gg_s k$ we also know that $k \notin Ch_s(I'' \cup \{i,j\})$. The irrelevance of rejected contracts property then ensures that $i \in Ch_s(I'' \cup \{i,j\}) = Ch_s(I' \cup \{i,j\})$, and we are done. □

Example A.2. Let $I = \{i,j,k\}$ and $S = \{s\}$. The preferences of s are given by

$$\{i,j,k\} \succ_s \{i,k\} \succ_s \{i\} \succ_s \{j\} \succ_s \{k\},$$

and there are no feasibility constraints ($\mathcal{F}_s = 2^I$). Here $i \gg_s j$ and $j \gg_s k$, but $i \not\gg_s k$.

A.2 Unique Stable Allocation that Admits Strong Justified Envy under Non-Substitutable Preferences

Example A.3. Let $I = \{i,j,k\}$, $S = \{s\}$, and $T = \{A,B,C\}$. To ease notation, we will represent the contract (i,s,t) by i_t, and similarly for all other students. The school's preferences are given by

$$\{j_A,i_B,k_B\} \succ_s \{j_A,i_C,k_B\} \succ_s \{j_A,k_B,i_A\} \succ_s \{j_A,i_B\} \succ_s \{j_A,i_C\} \succ_s \{j_A,k_B\} \succ_s \{j_A,i_A\} \succ_s \{j_A\} \succ_s \{i_B,k_B\} \succ_s \{i_B,j_B\} \succ_s \{i_C,k_B\} \succ_s \{i_C,j_B\} \succ_s \{k_B,j_B\} \succ_s \{k_B,i_A\} \succ_s \{j_B,i_A\} \succ_s \{i_B\} \succ_s \{i_C\} \succ_s \{k_B\} \succ_s \{j_B\} \succ_s \{i_A\} \succ_s \emptyset.$$

These preferences satisfy the law of aggregate demand (Hatfield and Milgrom, 2005), bilateral substitutability (Hatfield and Kojima, 2010), and substitutable completability (Hatfield and Kominers, 2015). Substitutable completability and bilateral substitutability are (independent) weaker conditions than substitutability, but they still ensure the existence of a stable allocation. Both conditions are also weaker than Hatfield and Kojima's (2010) unilateral substitutability (Kadam, 2017).

To verify that Ch_s satisfies substitutable completability, consider the following substitutable completion of $\succ_s$ (underlined allocations violate the constraint that each student can only be assigned one contract):

$$
\begin{aligned}
&\underline{\{j_A, i_B, i_C\}} \bar{\succ}_s \{j_A, i_B, k_B\} \bar{\succ}_s \underline{\{j_A, i_B, j_B\}} \bar{\succ}_s \underline{\{j_A, i_B, i_A\}} \bar{\succ}_s \{j_A, i_C, k_B\} \bar{\succ}_s \\
&\underline{\{j_A, i_C, j_B\}} \bar{\succ}_s \underline{\{j_A, i_C, i_A\}} \bar{\succ}_s \underline{\{j_A, k_B, j_B\}} \bar{\succ}_s \{j_A, k_B, i_A\} \bar{\succ}_s \underline{\{j_A, j_B, i_A\}} \bar{\succ}_s \{j_A, i_B\} \bar{\succ}_s \\
&\quad \{j_A, i_C\} \bar{\succ}_s \{j_A, k_B\} \bar{\succ}_s \underline{\{j_A, j_B\}} \bar{\succ}_s \{j_A, i_A\} \bar{\succ}_s \{j_A\} \bar{\succ}_s \underline{\{i_B, i_C\}} \bar{\succ}_s \{i_B, k_B\} \bar{\succ}_s \\
&\quad \{i_B, j_B\} \bar{\succ}_s \underline{\{i_B, i_A\}} \bar{\succ}_s \{i_C, k_B\} \bar{\succ}_s \{i_C, j_B\} \bar{\succ}_s \underline{\{i_C, i_A\}} \bar{\succ}_s \{k_B, j_B\} \bar{\succ}_s \{k_B, i_A\} \bar{\succ}_s \\
&\quad \{j_B, i_A\} \bar{\succ}_s \{i_B\} \bar{\succ}_s \{i_C\} \bar{\succ}_s \{k_B\} \bar{\succ}_s \{j_B\} \bar{\succ}_s \{i_A\} \bar{\succ}_s \emptyset.
\end{aligned}
$$

The completed preferences $\bar{\succ}_s$ can be thought of as follows: the school always chooses j_A when it is available, and in addition chooses from the remaining contracts according to a "responsive" choice function (in quotation marks as it allows choosing two contracts with the same student) with a capacity of 2 and the order over singletons $i_B > i_C > k_B > j_B > i_A$, and all other contracts (j_C, k_A, and k_C) are unacceptable.

Next, note that $i \gg_s j$: if j_A got accepted and i_A is also available, then either it or another contract with i is chosen. If j_B got accepted and i_B is also available, then i_B is also chosen. j_C is unacceptable, and so is never chosen. Finally, if we look at $X' = \{k_B\}$, then $i_B \in Ch\,(X' \cup \{i_B, j_B\})$, but $j_B \notin Ch\,(X' \cup \{i_B, j_B\})$.

Let the students' preferences be given by

$$
\begin{aligned}
i &: i_A \succ_i i_C \succ_i \emptyset \\
j &: j_B \succ_j j_A \succ_j \emptyset \\
k &: k_B \succ_k \emptyset.
\end{aligned}
$$

One can easily verify that the unique stable allocation is $\{i_C, j_A, k_B\}$, and under it i has strong justified envy toward j.